2. "The Malfatti problem" by Professor F. W. OWENS, Pennsylvania State College.

3. "On matrix equations" by Professor ARNOLD DRESDEN, Swarthmore College.

4. "Space filling polyhedra" by Professor A. H. WILSON, Haverford College.

5. "Difference equations" by Professor TOMLINSON FORT, Lehigh University.

6. "Positive integral solutions of an indeterminate equation" by Professor R. MORRIS, Rutgers College.

1. Professor Crawley's talk on "Descartes' Geometry" was a brief sketch of the contents of this book, limited by the time available to a description of Descartes' method of attack of the problem in which he first introduces the use of co-ordinates, and a short reference to his application of his method to the problem of constructing a normal to a curve at a given point.

2. In this paper, Professor Owens gave a brief historical sketch of the Malfatti problem, i.e. the problem of constructing in a triangle three circles each one tangent to two sides of the triangle and to the other two circles. He concluded with an original solution somewhat different from any usually given.

3. This paper consisted of a brief report on a method recently developed by Mr. W. E. Roth of the University of Wisconsin for the determination of solutions of the equation $P(X)=A$, which are expressible as polynomials in A, where $P(\lambda)$ is a polynomial in λ with scalar coefficients and without constant term, A is a matrix of order n and X is to be determined as a matrix of order n. The method is very elementary in character and is applicable in many instances where the matrix A is singular. It has the further advantage over methods developed by earlier writers in not requiring the use of elementary divisors. Mention was also made of another method developed by Professor M. H. Ingraham.

4. Professor Wilson discussed first the analogous problem for two dimensions: to cover the plane by repetitions of a polygon. He enumerated the well known classes of polygons which have this property and gave an account of the work of MacMahon, showing charts of some of the repeating polygons developed by him.

For space of three dimensions, in addition to the prisms, the classical space filling polyhedra, the rhombic dodecahedron, the bees'-cell, and the tetrakaidecahedron were described. Models of these were shown and a brief sketch of their significance in nature was given, with especial reference to the work of the Harvard Medical School in determining the form of certain cells. He concludes with an account of the work of Sommerville, who solved the problem of finding all plane-filling triangles and all space-filling tetrahedra for non-

Euclidean geometry as well as Euclidean geometry. Models of the four tetrahedra for Euclidean space were shown.

5. Professor Fort outlined the history and literature of difference equations and stated the principal problems involved. In particular, he dwelt upon the advances in difference equations which have been made in America.

6. Professor Morris discussed a problem in Diophantine analysis which arose as a generalization of a problem recently proposed for solution in this journal.

J. R. KLINE, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-second regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at Georgetown University, Washington, D. C., on Saturday, December 3, 1927. Two sessions were held, one in the morning and one in the afternoon, Professor J. R. Musselman, Chairman of the Section, presiding at both sessions.

Sixty-eight persons attended the meeting, including the following forty-six members of the Association: O. S. Adams, L. M. Blumenthal, C. C. Bramble, J. A. Bullard, J. F. Burley, Helen Calkins, Paul Capron, G. R. Clements, A. B. Coble, A. Cohen, Tobias Dantzig, L. S. Dederick, Alexander Dillingham, J. A. Duerksen, Harry English, P. J. Federico, G. L. Fentress, Michael Goldberg, Harry Gwinner, W. M. Hamilton, P. E. Hemke, L. S. Hulburt, H. P. Kaufman, L. M. Kells, W. D. Lambert, A. E. Landry, C. L. Leiper, T. W. Moore, W. K. Morrill, F. D. Murnaghan, J. R. Musselman, J. W. Peters, E. C. Phillips, Otto Ramler, C. H. Rawlins, Jr., J. N. Rice, A. W. Richeson, H. M. Robert, Jr., R. E. Root, W. F. Shenton, T. H. Taliaferro, John Tyler, Paul Wernicke, J. E. Willis, Elizabeth Wilson, E. W. Woolard.

At the beginning of the morning session, Rev. Charles W. Lyons, S. J., President of the University, made a short address of welcome to the members and guests of the Section. During the intermission between the morning and afternoon sessions those attending the meeting were entertained at luncheon by the University. Just before the formal opening of the afternoon session Professor Cohen, of Johns Hopkins University, as Maryland Representative of the American Mathematical Society, made a short informal address reminding those present that the Society and the Association were making an experiment in holding their annual meetings so far south as Nashville, and suggested that all those who possibly could should avail themselves of the opportunity thus afforded to attend these joint meetings with that of the A. A. A. S. A good attendance, incidentally, would convince the Society and the Association that their experiment was sympathetically received. At the close of the afternoon

session the Executive Committee held a brief meeting at which it was decided to hold the 1928 meetings as follows: the Summer Meeting on Saturday, May 5, at Annapolis, Maryland, and the Winter Meeting on Saturday, December 1, at Baltimore, Maryland.

The following eight papers were presented:

1. "On the equilibrium of an elastic blade" by Dr. TOBIAS DANTZIG, University of Maryland.

2. "A configuration of ten points analogous to the Desargues configuration" by Professor A. B. COBLE, Johns Hopkins University.

3. "An extension of the Gauss problem of eight queens" by Dr. L. M. BLUMENTHAL, Johns Hopkins University.

4. "Observers' patterns" by HOWARD S. RAPPLEYE, U. S. Coast and Geodetic Survey. (By invitation.)

5. "Rational paths with supersingular equations" by Rev. E. C. PHILLIPS, Georgetown University.

6. "Some remarks on a recent textbook of College Geometry" by Dr. J. N. RICE, Catholic University of America.

7. "On the application of differentials and definite integrals in physics" by Mr. E. W. WOOLARD, U. S. Weather Bureau.

8. "Some remarks on the danger of hasty generalization" by Professor R. E. ROOT, Postgraduate School, U. S. Naval Academy.

The authors' abstracts of their respective papers follow in the same order as enumerated above.

1. In the classical treatment of the beam problem certain assumptions are made which make it possible to reduce the problem to a linear differential equation of the second order, which gives a solution in finite form. These approximations become untenable in the case here considered. The author shows that the problem depends on the solution of the differential equation $(d^2\phi)/(ds^2)=s\cos\phi$ and seeks a power expansion which satisfies this equation. He considers the most general holomorphic expression of the form $y''=x^\tau f(y)$ and shows that it admits a power expansion in terms of x^{m+2}. Applying this theorem he finds a series solution for the elastic line. With excellent approximation this can be approached by a quartic parabola with vertex at the free end. The locus of the free end as the length of the strip changes is sought. A practical application is the determination of the elastic modulus of a material and method for the measure of stiffness. The method used in this paper is applicable to the case of other modes of loading and support.

2. If four nodes of a rational plane sextic are isolated, the four pairs of nodal parameters and the six pairs of parameters of points in which the six lines joining the four nodes meet the sextic again furnish us a set of ten specially

related binary quadratics. If these quadratics are plotted as points with reference to a norm-conic a ternary configuration of ten points is obtained which contains ten triangles each perspective to a triangle whose sides are defined by the ten points. This configuration is analogous to that of Desargues but contains nine, rather than three, absolute constants.

3. The problem of ascertaining the number of ways in which eight queens may be placed on a chess board so that no queen can capture any other was proposed to Gauss by Nauck. In this paper is considered an extension of the problem to the case of n queens on a board of n^2 squares. It is shown that whenever n is not a multiple of two or three many solutions may be written down at once. These solutions are generated by a key solution. The problem is seen to be equivalent to one in combinatory analysis.

4. In the work of first-order leveling as carried on by the U. S. Coast and Geodetic Survey, the rods are read by estimating tenths of centimeter graduations. Each rod is read at three points. The height of the instrument and consequently the height at which the rods are read being purely accidental, there should be about an even distribution among the ten digits in the resulting estimated millimeters. This paper is a preliminary statement of results obtained by tabulating over 30,000 separate estimations of millimeters.

The results were grouped for different observers and for different observing conditions. The diagrams or "patterns," by means of which the results were shown, displayed some startling differences between the work of different observers and even for the same observers under different observing conditions.

The possibility of devising a test to determine the fitness of a particular observer for this class of work, before taking the field with a level party, was discussed. Certain other practical results, such as a limit on the length of sight which may possibly arise from this investigation, were noted.

5. The main purpose of this paper was to call attention to some advantages of the parametric equations of a plane curve over the usual Cartesian equation, especially when dealing with curves considered from the kinetic point of view as geometric loci traced by a moving point. As a simple example the parabolic trajectory of a bullet was cited. When the angle of elevation is 90°, the Cartesian equation degenerates or breaks up into linear factors ($x^2=0$). Such an equation gives very inadequate and in a certain sense misleading information as to the actual trajectory, whereas the parametric equations even in this special case still give us full and exact information. It was shown that the degeneration of the Cartesian equation results from imposing on the trajectory the condition of having a cusp, which is requiring the curve to be supersingular. Any path which, like the trajectory of the bullet, is an unambiguous, unicursal locus of finite order, whose Cartesian coordinates can be represented by rational

functions of a single real variable parameter, was called a rational path; it was shown that the parametric equations of such a path of order n can be made to satisfy the necessary and sufficient conditions for the existence of 2n–2 (or fewer) cusps, whereas a proper curve of order n can have at most $(3n-6)/2$ cusps; hence such loci may be called supersingular rational paths.

6. This paper consisted of remarks on Altshiller-Court's *College Geometry* based on class-room experience with the text.

7. Attention was drawn to certain loose statements about differentials frequently encountered in textbooks, as illustrated by the customary derivation of the hypsometric formula; and a rigorous derivation of this formula, making use of Duhamel's theorem, was outlined. It was particularly emphasized that if $y=f(x)$, dy is not, in general, the increment in y produced by the increment dx in x; that neither dy nor dx is "infinitely small"; and that $dy \neq \lim \Delta y$, $dx \neq \lim \Delta x$.

8. In the remarks by Professor Root, attention was called to the fact that, whilst mathematics is a deductive science, yet the methods and devices of inductive reasoning are very properly and effectively used to aid invention and discovery in mathematical fields. In expository and text book writing, analogy and generalization are frequently employed, the essential deductive proofs often being replaced by such rhetorical devices as "it is evident that," or "similarly it may be shown," etc. The dangers of these devices was shown by an example drawn from a text book in mechanics, the error arising from an incorrect use of analogy between the ellipse of inertia in a plane and the ellipsoid of inertia in space.

E. C. PHILLIPS, *Secretary*

THE ANALYSIS SITUS OF THE PLANE WHEN THE DIRECTED LINE IS TAKEN AS ELEMENT[1]

By JESSE DOUGLAS[2]

In previous papers on the geometry based on the line as element,[3] the writer has made use of a certain representation of the oriented lines of the plane on the points of a cylinder, which he has since found had already been used by Blaschke.[4] In the present paper, this and related representations are employed to study the analysis situs of the plane regarded as a manifold of oriented lines.

[1] This paper is a development of one presented to the American Mathematical Society, Oct. 28, 1922. Extension to space is made in §§5, 6.

[2] National Research Fellow in Mathematics.

[3] Abstracts are in the Bulletin of the American Mathematical Society, vol. 26 (1920), p. 437; vol. 27 (1921), p. 398.

[4] *Über einige unendliche Gruppen von orientierten Berührungstransformationen in der Ebene*, Mathematische Annalen, vol. 69 (1910), pp. 204-217; and *Über die Laguerresche Geometrie orientierter Geraden in der Ebene*, Archiv der Mathematik und Physik, vol. 18 (1911), pp. 132-140.

We shall use the term *spear* (due to E. Study) as alternative to oriented line. Let the spears σ of the plane be referred to (Hessian) coordinates (ω, p) as follows. Regard the plane as xy-plane of a Cartesian coordinate system in space, and let τ denote the spear passing through the origin O, perpendicular to σ, and oriented relative to σ as the positive x to the positive y axis. If M is the intersection of τ with σ, we define p as the algebraic length of the directed segment OM. Thus p will be positive or negative according as σ indicates positive or negative rotation about O. ω shall denote the angle made by τ with a fixed spear α through O; ω is determined up to the addition of a multiple of 2π, and its trigonometric functions are therefore uniquely determined. Then we take as representing cylinder the one of equation $x^2+y^2=1$, and make correspond to σ the point $(x, y, z)=(\cos \omega, \sin \omega, p)$.

A *beam* of spears will be the name given to the totality of spears parallel, including sense, to a given one. It is evident that in the representation just defined the spears of a beam correspond to the points of an element of the cylinder.

We shall find that the analysis situs of the plane regarded as a manifold of spears depends essentially on the conventions made relative to elements at infinity.

1. Cylinder. Suppose we take only the lines at finite distance in the plane (metric plane) and make each into two spears by giving it its two possible

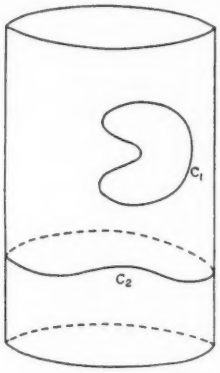

FIG. 1

orientations. The resulting spear manifold M_1 is by the above representation in one-one continuous correspondence with the points at finite distance of the cylinder. *The analysis situs of the metric plane of spears is identical with that of the metric cylinder regarded as a point manifold.*

The outstanding implication of this is that there are precisely two distinct types of simple, continuous, closed spear series in the plane non-equivalent as to one-one continuous transformation. These correspond respectively to the two cylindrical curves C_1 and C_2 in Fig. 1. Since it is characteristic of C_1 that upon it the total variation of ω ($\int_{c_1} d\omega$) is zero, while upon C_2 this total variation is 2π, we may distinguish the two types of closed spear series by saying that in the one the total rotation of the moving spear is zero, in the other, 2π. Any two simple, continuous, closed spear series both of the first or both of the second type are equivalent in the analysis situs sense, but never a series of the first type to one of the second. In illustration of the first type we may use Fig. 2, which, though open as a point locus, is closed as a spear locus. Typical for the second type is a pencil or oriented circle.

FIG. 2

As a second, curious, inference from our representation, we cite the following. Let a point move continuously so as to surround the cylinder twice and return to its original position. It is evident that it cannot do this without crossing its own path (Fig. 3). Translated to the plane, this signifies that if a spear move continuously through a total rotation of 4π in returning to its

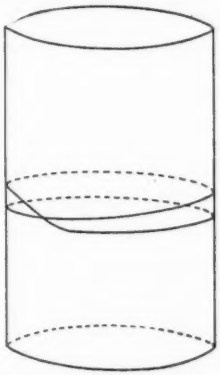

FIG. 3

original position, there must be some position which is occupied more than once by the moving spear. This is not nearly so clear to the intuition in the plane manifold of spears as it is in its equivalent form on the cylinder.

2. Projective cone. Imagine that we subject the space containing the cylinder to a projective transformation so that the cylinder becomes a cone.

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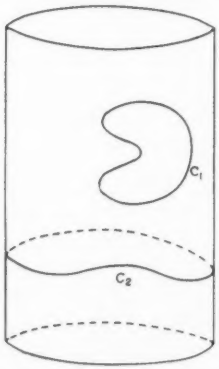

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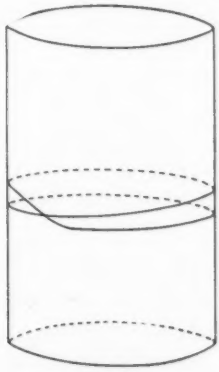

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2. Projective cone. Imagine that we subject the space containing the cylinder to a projective transformation so that the cylinder becomes a cone.

The cylinder being regarded from the metric point of view, and the cone considered as including its conic at infinity through which it closes upon itself, the correspondence between the two surfaces is one to one and continuous, except that to the vertex of the cone there corresponds no point of the cylinder, and therefore no spear of the plane. But if a point on the cone is near to the vertex, it corresponds evidently to a spear at great distance from the origin O and oriented positively or negatively according as the point belongs to the one or the other nappe of the cone.

This impels us to adjoin to the spear manifold M_1 described in §1 an element at infinity in the form of the line at infinity of projective geometry, to which however, unlike the other lines, we attach no orientation. We define the neighborhood of a spear at finite distance in the usual way, and regard the neighborhood of the element at infinity as consisting of the spears of either positive or negative orientation as to O which are at great distance from O. The manifold M_2 so defined is homeomorphic with the projective cone, the line at infinity corresponding to the vertex of the cone.

A representation by means of a surface located altogether in the finite part of space is illustrated by Fig. 4.

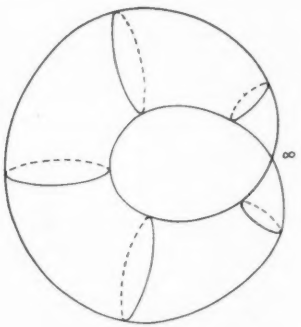

FIG. 4

3. Sphere. Consider the sphere $x^2+y^2+z^2=1$ in relation to the cylinder $x^2+y^2=1$. We can represent the one on the other by means of rectilinear rays through the origin. If we delete temporarily the north and south poles of the sphere, the correspondence is one-one and continuous. To points near the north pole correspond spears of great distance from O and positive orientation as to O; to points near the south pole spears of great distance from O and negative orientation as to O.

Thus, in order to obtain a correspondence which shall be without exception one-one and continuous, we must adjoin to the manifold M_1 of §1 two spears at infinity, one of positive, one of negative orientation as to the finite part of the

plane. This is done by using the projective plane and giving to the line at infinity as well as to every other line its two possible orientations. The analysis situs of the so constructed spear manifold is simply that of the sphere (genus zero).

4. Torus. The fourth and most interesting analysis situs is arrived at as follows. Start with a straight line and a circle tangent to it at a point T; then remove from the circle the point O diametrically opposite to T. Denoting the line regarded as metric line (no point at infinity) by λ, and the deleted circle by γ, we then set up a one-one continuous point correspondence between λ and γ by mating the points of λ and γ respectively which lie on the same straight line through O (Fig. 5).

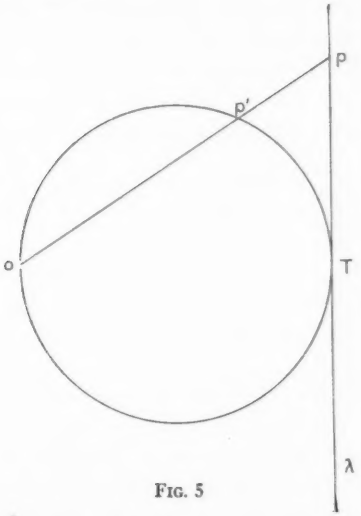

FIG. 5

If now the entire figure be revolved about an axis in its plane parallel to λ, and on the side of λ opposite to the circle, λ generates a cylinder, and γ a torus from which the outer equator has been removed; and we have furthermore a homeomorphism between this deleted torus and the cylinder, and therefore a homeomorphism between the deleted torus and the spear manifold M_1.

It is natural to restore to the torus its missing outer equator and to preserve the homeomorphic character of the correspondence between spears of the plane and points of the torus by inventing appropriate improper spears.

To this end, we recall that each beam of spears corresponds to an element of the cylinder, which corresponds in turn to a meridian of the torus, and that points near the missing equator correspond to spears at great distance from the origin. This indicates that we postulate in each beam a spear at infinity, and

regard the infinite spears of different beams as distinct. A neighborhood of such a spear at infinity will consist of those spears at great distance from the origin which differ little in direction from this spear at infinity.

The resulting manifold M_4 of spears is homeomorphic with the torus (genus one).

Since every continuous closed (not necessarily simple) curve on the torus is, if not reducible[1] to a point, equivalent[1] to a certain finite or discretely infinite number of parallels together with a similar number of meridians, it follows that every continuous closed spear series, not of the type of Fig. 2, is equivalent to a discrete number of pencils together with a discrete number of beams.

As a preliminary to the extension to space to be made in §6, we remark that instead of the torus we may use the two-dimensional manifold of ordered point-pairs (P, Q)[2] of a circle as the homeomorphic equivalent of the spear manifold M_4.

Extension to Space.

5. Oriented planes or "pages." The preceding considerations may be extended to space in two ways, according as we regard space as a three-dimensional manifold of oriented planes or a four-dimensional manifold of spears.

Out of each plane we may construct two oriented planes, or, as we shall call them, *pages*, by distinguishing in a preferential way one of the sides of the plane from the other. The page will be said to *face* in the direction of the preferred side of the plane.

For page coordinates we adopt the following. Let a spear be drawn through the origin O of Cartesian coordinates perpendicular to the plane of the given page and oriented so that the page faces in the sense of the arrow-head. Let l, m, n, with $l^2+m^2+n^2=1$, denote the direction cosines of this spear, and p the plus or minus value of the directed segment described by a point moving along the spear from O to its piercing point with the plane. A homeomorphism between the pages of metric space and the points of the metric hypercylinder $x^2+y^2+z^2=1$ in four-dimensional space (x, y, z, w) is established by causing to correspond to the page (l, m, n, p) the point $(x, y, z, w)=(l, m, n, p)$.

As in two dimensions, we can replace the hypercylinder by a hypercone, a hypersphere, or a hypertorus, by the appropriate conventions as to elements at infinity.

6. Spears. Here, as before, the analysis situs depends entirely on the postulates adopted relative to elements at infinity. Of the many different types that

[1] Since we do not require the curve to be simple, we here understand "reducible" and "equivalent" in the sense of one-many continuous transformation.

[2] Q is allowed to be identical with P.

are possible,[1] we shall confine ourself to the one which results from the assumption that in each beam of spears there is a unique infinite spear, distinct from that of any other beam. It is to be remarked that this is quite different from the usual assumption of projective geometry. There a beam is simply a sheaf whose vertex is in the plane at infinity, so that, whether we deal with oriented or unoriented lines, there are evidently ∞^1 different elements at infinity in each beam. For example, in our convention there is exactly one spear at infinity directed vertically upward, and ∞^1 horizontal spears at infinity; whereas in projective geometry these two numbers are reversed.

It is readily shown that just as for the corresponding convention in the plane (§4) the analysis situs is that of the ordered point-pairs of a circle, here the analysis situs is that of the ordered point-pairs of a sphere.

To see this, let us consider all the spears of any given beam β. Having chosen a fixed sphere, center O, let us denote by P the point in which the spear belonging to β and passing through O pierces the sphere, in emerging from it. P will be the first point of the ordered point-pair corresponding to any spear σ of β.

The second point Q is determined as follows. Let π be the plane through O perpendicular to the direction of the beam β. We regard π as an inversion plane, and make correspond to each proper spear σ of β the point in which σ pierces π, and to the improper spear of β the unique infinite point of π. When we follow this representation of β on π by a stereographic projection of π on our fixed sphere, using P as pole, there is made to correspond, in a one-one continuous way, to each spear σ of β a point Q of the sphere. This Q shall be taken as second point of our ordered point-pair.

The correspondence $\sigma \rightarrow (P, Q)$ is evidently homeomorphic. The pairs (P, P) whose two points coalesce correspond to the improper spears of space.

A representation of our four-dimensional spear manifold by means of a four-spread in euclidean five-space can be had by starting with an (ordinary) two-sphere together with a non-intersecting two-flat, both located in the same three-flat of the five-space; then rotating the two-sphere about the two-flat freely in the five space (∞^2 positions). The manifold of points formed by the resulting four-spread, torus-like in a two-dimensional way,[2] is easily set into homeomorphism with the ordered point-pairs of a sphere, and therefore with the spears of space.

[1] Besides the one considered in this section there are the following: no elements at infinity; one such element; the lines at infinity of projective geometry, unoriented; the same, each given its two possible orientations.

[2] Not, of course, to be confused with the four-manifold (analogous in one higher dimension to the hypertorus of §5) which is obtained by rotating a three-sphere about a non-intersecting three-flat, both located in the same four-flat of our five-space. The resulting four-manifold may be said to be torus-like in a one-dimensional way (∞^1 positions of the moving three-sphere).

NAPIER'S LOGARITHMS AS HE DEVELOPED THEM

By W. D. CAIRNS, Oberlin College

In 1614 Napier published his *Mirifici Logarithmorum Canonis Descriptio* which incorporates his tables and in 1619 his *Mirifici Logarithmorum Canonis Constructio* which explains the method by which he formed his tables. Various editions and translations from the Latin originals are to be found, such as the combined edition of 1620, Edward Wright's 1616 translation of the *Descriptio*, the 1807 re-printing of the 1614 *Descriptio* by Maseres in the sixth volume of his *Scriptores Logarithmici*, and the 1889 translation of the *Constructio* by W. R. Macdonald. Reference may also be made to the facsimile reproduction of the title page and the first eleven pages of Wright's 1616 translation of the *Descriptio* in the Napier Tercentenary Memorial Volume (Longmans, Green and Co., 1915). Yet numerous colleges and universities have not been able to obtain material on Napier's logarithms suitable for their purposes; and it is the purpose of this paper to give the essentials of his treatment. The details of his computations throw an interesting light on the status of mathematical practice at the beginning of the seventeenth century. We leave it to the historians, however, to determine how much of this is due to Napier in particular.

Noting that he purposes to avoid all multiplications, divisions, and the more difficult extractions of roots and that his method depends on a correspondence between an arithmetic and a geometric progression, he proceeds to compute the latter as simply as possible, viz., by subtracting successively a simple fractional part. Thus, using 10^7 (in modern notation) as "radius," he takes the ratio in his "First Table" as .9999999, and one ten-millionth is subtracted successively one hundred times; the last number in this table, to the seven decimal places which he preserved, is 9999900.0004950. Replacing this last number by 9999900, the "Second Table" begins with 10^7, uses the ratio .99999 by subtracting one hundred-thousandth fifty times in succession; the last number in the table is 9995001.222927, a result slightly incorrect, for the decimal part should be .224804, as noted by Macdonald. Again, using 9995000 as "sufficiently near to" this last number, the "Third Table" begins with 10^7, uses the ratio .9995 by subtracting one two-thousandth twenty times in succession, the last number being 9900473.57808; this set of 21 numbers forms the first of 69 columns in this third table. The "easiest" number nearest to the last number, viz., 9900000, is taken as the first number in the second column, and in accordance with this, the first numbers in the 69 columns are formed, beginning with 10^7, by subtracting one hundredth 68 times in succession, that at the head of the 69th column being recorded by Napier as 5048858.8900. Next he fills in the other numbers in the 69 columns, row after row, by reducing those

in the first column by one-hundredth successively; for example, the last (twenty-first) row is

$$9900473.5780, 9801468.8423, 9703454.1539, \cdots, 4998609.4034.$$

This last number, he notes, is practically half the radius, and he has thus computed a whole series of geometrical means between the radius and the half radius with the ratio .9995, and can insert 49 means between each pair of consecutive numbers by the numbers of the second table used as multipliers, and can further insert 99 means between the resulting numbers by means of the numbers of the first table. His choice of the base 10^7 is now justified by the fact that the numbers thus obtained differ consecutively by no more than unity and that the decimal parts may be neglected.

At this point in Napier's explanations he introduces his theory of logarithms, centering in the definition quoted here from the *Constructio*; the numbers of the sections are given for convenience in reference:

"26. The logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which radius began to decrease geometrically, and in the same time as radius has decreased to the given sine." The reader must recognize that to Napier the sine was a line, or the number measuring the line, as in the present-day line representation of functions of angles. Various theorems follow which are needed for the development of his tables, several of these being given here in substance.

27. Zero is the logarithm of the radius.

29. An upper and lower limit are derived from the definition in section 26. " $\cdots$ the given sine being subtracted from radius the less limit remains, and radius being multiplied into the less limit and the product divided by the given sine, the greater limit is produced $\cdots$."

33. By section 29 the first "proportional" of the first table, 9999999, has its logarithm between the limits 1.0000001 and 1.0000000, that of the second proportional, 9999998.0000001, between 2.0000002 and 2.0000000, etc. Hence these logarithms may be taken as 1.00000005, 2.00000010, etc.

36. "The logarithms of similarly proportioned sines are equi-different." This is proved by the definition of logarithms, also by a comparison of the upper and lower limits.

41. The method for finding the logarithms of sines or "natural numbers" not found in the first table will be shown clearly enough by his examples. Let the given sine be 9999975.5000000; the nearest sine in the table is the twenty-sixth number, 9999975.0000300, and is smaller than the given sine. By section 33 the limits of the logarithm of this last are 25.0000025 and 25.0000000. By the same method used in proving the rule in section 29, he proved that the difference of the logarithms of these two sines has the limits .49997122 and .49997124, whence to seven decimal places the required limits of the logarithm

are 24.5000313 and 24.5000288, and he uses 24.5000300. Similarly the limits for 9999900.0004950, the last sine in the first table, which are 100.0000100 and 100.0000000, yield 100.0005050 and 100.0004950 as the limits for the logarithm of 9999900, the second number in the second table, and from this the logarithms for all the other numbers in the second table are found by the rule for proportionals.

43. For the method of finding the logarithms of sines not found in the second table, an example will suffice. Let the given sine be 9995000, the second number of the first column in the third table. The nearest sine in the second table is the last, 9995001.222927, the limits for its logarithm being 5000.0252500 and 5000.0247500. He finds "a fourth proportional, which shall be to radius as the less of the given and table sines is to the greater." This he does "by multiplying the difference of the sines into radius, dividing this product by the greater sine, and subtracting the quotient from radius." By section 36 the logarithm of this fourth proportional differs from that of the radius by as much as the logarithms of the given and table sines differ from each other. He then finds the limits of the logarithm of the fourth proportional by aid of the first table, and adds them to or subtracts them from the limits of the logarithm of the table sine, to obtain the limits of the logarithm of the given sine. In this example, the fourth proportional is 9999998.7764614, and the limits of its logarithm, found by 41, are 1.2235387 and 1.2235386. Adding these to the former limits, he has 5001.2487888 and 5001.2482886 as the limits of the required logarithm, which is therefore taken (midway) between these, 5001.2485387.

44. From this the logarithms of all the other numbers in the first column of the third table are found by the rule for proportionals.

45, 46. It only remains to find by the same method the logarithm of 9900000, the first number in the second column, viz., 100503.3210291, after which the logarithms of all the numbers of the third table are found either by adding this last logarithm successively along any row, beginning with the first column, or by adding 5001.2485387 successively down each column. The logarithm of the last entry, 4998609.4034, is thus found to be 6934250.8007528.

47. The "Radical Table," next formed by Napier, merely enters all the numbers of the third table and their logarithms in 69 double columns, the "natural numbers" or sines to four decimal places, the logarithms to one; and from this he compiled his logarithm tables. For example, in getting the logarithm of 7489557, the nearest table sine being 7490786.6119, he finds the difference of these two to be 1229.6119. Multiply this by the radius and divide by "the easiest number," which may be either of the foregoing or the simpler 7490000, the result being 1640.1; this added to 2889111.7, the logarithm of the table sine, gives 2890752. Similarly for the logarithms of all sines within the limits of the radical table.

51, 52. By this same rule the logarithm of 5000000 is 6931469.22, of 8000000 is 2231434.68, and by a further use of the same rule the logarithm of 1000000 is 23025842.34.

54. These particular logarithms and the rule for proportionals give him the logarithms of all sines outside the table limits. For example, the logarithm of 378064, which is 1/20 of 7561280, a number in the limits of the table, is 32752756, the sum of 2795444.9 and 29957311.56, the logarithms of 7561280 and 20.

59. Napier's table is constructed in quite the same form as used at present, except that the second (sixth) column gives sines for the number of degrees indicated at the top (bottom) and of minutes in the first (seventh) column, the third (fifth) column gives the corresponding logarithm and the fourth column gives the "differentiae" between the logarithms in the third and fifth columns, these being therefore essentially logarithmic tangents or cotangents. Reproductions of some of these pages may be seen in Macdonald's translation and in Cajori's History of Mathematics.

Suffice it to say, in closing this note, that the second book of the *Descriptio*, "On the remarkable advantage (praeclaro usu) in trigonometry of the wonderful canon of logarithms," shows the solutions of a right triangle by logarithms in the manner familiar to us, the solutions of oblique triangles by the law of sines and of tangents, and of the case where three sides are given, the last by the expedient of drawing the altitude from the intersection of sides a and c and using an auxiliary isosceles triangle of which this is the altitude and side c is one of the equal sides; then follows the portion devoted to spherical triangles, wherein Napier uses his "circulares," known to us as Napier's circular parts, to solve quadrantal spherical triangles, and completes the theory of spherical triangles essentially in the way adopted at the present day.

ON THE ORIGIN OF THE TERM "ROOT." SECOND ARTICLE

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In a recent number of this Monthly[1] the writer suggested that the term "root" goes back to the Arabic *jadhr*, a word originally meaning a concrete number designating a geometric magnitude, as contrasted with an abstract number. He further referred to some passages of the algebra of al-Khowârismî which clearly defined *jadhr* as the side of a square multiplied into a square unit, and proposed that *jadhr* should not be translated as "root" but rather as "basis" or "foundation."

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[1] Vol. 33, pp. 261–265.

It is the purpose of this paper to inquire further into the same problem, to examine fully the algebra of al-Khowârismî, and to see how he understands this term *jadhr* in those numerous passages of his book in which the above definition cannot be applied. In connection with this special question the writer will attempt, in a short and necessarily incomplete sketch, to comment on the origin of the Arabic terminology of algebra in general.

An intensive study of al-Khowârismî's algebra shows that the Arabic mathematician does not use the word *jadhr* as a clear and unequivocal term, but that he combines with it different ideas and meanings. This fact was observed long ago by Rosen[1] in regard to the term *mâl* (square), and by the writer in regard to the term *Misâḥah*.[2] In the beginning of his book[3] we find the following definitions: "And I found that the numbers which are required in the operations of the algebra are of three classes, namely, *jadhr's*,[4] *mâl's*,[5] and the abstract number without reference to the *jadhr* or to the *mâl*. Of these the *jadhr* is any *shay'* which can be multiplied into itself, be it a unit, any number above a unit, or any fraction below it. The *mâl* is anything which is obtained as the result of the multiplication of the *jadhr* into itself. The abstract number is any number expressed in words[6] without reference to *jadhr* or *mâl*." The definition of the *jadhr* forming the basis of the other two definitions is rather obscure. It has two essential characteristics. There are the *shay'* and the *maḍrûb fi nafsihi*, "which can be multiplied into itself." Al-Khowârizmî himself does not tell us what the *shay'* is. But while he defines here the term *jadhr* by the term *shay'*, we find him in two other passages[7] defining the term *shay'* by the term *jadhr*, a procedure which, of course, does not make the definition any clearer. The same lack of clearness is seen in the second characteristic *maḍrûb fî nafsihi*, liable or capable of being "multiplied into itself."[8]

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[1] *Algebra of Mohammed ben Musa*, English text, p. 50, note. See also Ruska, *Zur ältesten arabischen Algebra*, pp. 47 et seq.

[2] This Monthly, vol. 34, p. 80.

[3] Arabic text, p. 3; English text, p. 5 et seq. The writer's translation, however, differs considerably from that given by Rosen.

[4] In the text, *judhûr*, the Arabic plural of *jadhr*.

[5] In the text, *amwâl*, the Arabic plural of *mâl*.

[6] Pure numerals.

[7] Arabic text, p. 15, lines 4–5; p. 16, line 15; English text, p. 21, 23. "I shall now teach you how to multiply the things (*ashyâ'*), that is to say, the roots (*judhûr*)"; "Ten less thing, the signification of thing (*shay'*) being root (*jadhr*)."

[8] There is also a philological obscurity in the formulation of it. *Maḍrûb* is the Arabic passive participle, which admits two meanings: *Multiplicatus* "multiplied," and *multiplicandus*, "to be multiplied" or "capable of being multiplied." Now al-Khowârizmî uses the same word *maḍrûb* in the definition of the *jadhr* and of the *mâl*. The *mâl* is the result of the *jadhr almaḍrûb fî nafsihi*, "of the root multiplied into itself." We, of course, have to give the author the benefit of the best interpretation, and to say that, in the definition of the *jadhr*, *maḍrûb* means *multiplicandus*, "to be multiplied"; while in the

The quality of "liable or capable of being multiplied into itself" does not belong merely to the *jadhr* or *shay'*. No one will deny that an abstract number or a square is also capable of being multiplied in itself.

But we get more and better information when we turn to the later Arabic mathematicians. Omar al-Khayyâmî (*c.* 1100) introduces his algebra[1] with a clear and philosophical exposition of its fundamental principles and notions. He says: Algebra deals with two main classes of numbers, the absolute or abstract numbers and the concrete numbers. The concrete numbers are the geometric magnitudes and unknown quantities. It is customary to call the unknown quantity *shay'*, its square *mâl*, the third power "cube," etc. The first power is also called by al-Khayyâmî, *jadhr* and *ḍil'*. The same opinion is also represented by al-Karkhî (*c.* 1020) who refers to the algebraic powers as the *ajnâs al-majhûlât* or *marâtib al-majhûlât*[2] the classes or degrees of the unknown quantities. This theory is still more clearly stated in the arithmetic of al-Qalaṣâdî (*d.* 1477–1486) who says[3]: "Algebra is based upon three classes, namely, the [absolute] numbers, the things (*shay's*) and the squares (*mâl's*). Some people add also the cubes. The [absolute] number has no basis (*ass*), and the basis of the *shay's* is one, and the basis of the *mâl's* is two, and the basis of the cubes is three. And among these classes nothing is known save the number. *Shay'* and *jadhr* are the same thing, i.e., the name for the unknown quantity. Al-Qalaṣâdî was the commentator of Ibn al-Bannâ (*c.* 1222) and we find his theory in the latter's work[4]: The basis of the *shay'* is one, the basis of the *mâl* two, etc.[5] The same theory of the basis (*ass*) is also given in the *Muqaddamat* (Prolegomena) of Ibn Khaldûn (born 1332).[6]

This gives us the clue to the right understanding of al-Khowârizmî's definition of the *jadhr*. It is a *shay'*, a concrete number having a basis (x), and it can be multiplied into itself and produce a number of a higher power (x^2).[7] These are the two characteristics which distinguish the concrete numbers from the abstract numbers.

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definition of the *mâl* it is *multiplicatus*, "multiplied." As we shall see later, however, a great deal of confusion was created by the use of this inaccurate language.

[1] Ed. Woepcke, Paris, 1851, Arabic text, p. 4 et seq.; French text, p. 5 seq.

[2] Woepcke, *Extrait du Fakhri*, p. 48.

[3] *Kashf al-asrâr'an'ilm ḥurûf al ghubâr*, translated by Woepcke, Rome, 1859, in an Extrait des Atti dell' Accademia Pontificia de' Nuovi Lincei, vol. 12, 1859. The Arabic manuscript was recently edited in lithograph copies, *sine anno et loco*. The passage occurs in the French text, p. 51; Arabic text, part 4, 3rd chapter on algebra; it is p. 36 of the unpaged text.

[4] *Talkhis*, translated into French by Aristide Marre, Extrait des Atti dell' Accademia Pontificia de' Nuovi Lincei, vol. 17, Rome, 1865, p. 30.

[5] He has also the rule that the quotient has as its exponent the difference between the exponents, and the product has as its exponent the sum of the exponents.

[6] Ed. Beirut, 1886, p. 422; ed. Paris, 1858, vol. 1, pp. 215-216; French translation by de Slane, p. 248.

[7] The pure number, say 4, if multiplied into itself, will produce only the number 16 of the same power.

Ruska[1] states that one who translates *mâl* as "square" has to take *jadhr* as the first power, which will lead to a very great difficulty in the terminology, because "eine 'erste' Potenz ist für Altertum and Mittelalter ein Unding." But the writer frankly owns that he does take *mâl* as the square, *jadhr* as the first power, and that this alleged "Unding" (absurdity) is, in his opinion, clearly stated in the works of the mathematicians al-Khowârizmî, al-Khayyâmî, al-Karkhî, Ibn al-Bannâ, Ibn Khaldûn, and al-Qalaṣâdî. Furthermore, the writer thinks that the very name *jadhr* had originally the meaning "basis," "foundation" to designate the first power of the concrete number, like the word *ass* (or *uss*) used by Ibn al-Bannâ, Ibn Khaldûn, and al-Qalaṣâdî which admits no other interpretation than "basis or foundation" and was used to indicate the exponent of the algebraic powers.[2] This term *ass* is much older than the Arabic authorities quoted above. It is found already in the Hebrew encyclopedia of knowledge written by Savasorda[3] (*c.* 1120) who cites it as a well-known Arabic term for the powers of ten in the decimal system, and translates it by the Hebrew *ush* belonging to the same root.[4] Ibn Ezra (*c.* 1150) calls it in Hebrew *yesôd* and *môsad*.[5] It goes back to the Greek πυθμένες, *pythmenes*, bases.[6]

There were two main characteristics in the science of algebra. The one consisted in the introduction of the method of equations to find the unknown quantity. This characteristic was given expression in the Arabic name *al-jabr wa-l-muqâbalah* which is the origin of our "algebra" and means "to set up an equation."[7] The second characteristic consisted in the introduction of operations with concrete numbers, or geometric magnitudes and algebraic powers, and this was emphasized by the Greek name "logistic" (λογιστική) which means a science dealing with sensible concrete things, not only with absolute numbers.[8] Hence the Arabic term *shay'* for the basic quantity, corresponding to Greek logos (λόγος) "thing," *res*, *cosa*, *coss*, Egyptian *hau*.[9] This conception alone leads us to understand why al-Khowârizmî introduced the *'adad al-mufrad*, the abstract number, and al-Khayyâmî, the *'adad al-muṭlaq*, the

[1] Loc. cit., p. 63.

[2] See Woepcke, translation of al-Qalasâdî, p. 10, note; p. 51, note 2; translation of a chapter of the *Prolegomena* of Ibn Khaldûn, p. 6, note 6. That *jadhr* is the first power was stated by Rosen, loc. cit., p. 6, note, and Woepcke, translation of al-Qalasâdî, p. 49, note 1.

[3] On this unpublished encyclopedia see Steinschneider, *Gesammelte Schriften*, pp. 388–404. Our quotation occurs *ibid.*, p. 394, note 10.

[4] The same form seems to occur also in the *Miqdash Me'aṭ* of Moses da Rieti, p. 14b. The passage reads: *Me'uqqab, meguddar we ishiyyi*, meaning a cube, square, and basic (number or magnitude). Read *ushiyyi* instead of *ishiyyi*. The *Miqdash Me'aṭ* is a Hebrew imitation of Dante's *Divina Commedia*, written about 1416.

[5] *Sefer ha-Eḥad*, p. 14; *Sefer ha-Mispar*, pp. 5, 6, 79; Steinschneider, loc. cit., p. 483.

[6] See Woepcke, translation of al-Qalasâdî, p. 51, note 2; Heath, Greek *Mathematics*, vol. 1, p. 55.

[7] See this Monthly, vol. 33, pp. 437 et seq.

[8] See Heath, *Greek Mathematics*, vol. 1, pp. 14 et seq.

[9] See Smith, *History of Mathematics*, vol. 2, p. 392 et seq.

absolute number, as contrasted with the *jadhr* and *mâl*. If *jadhr* and *mâl* were also only abstract roots and squares, as Ruska[1] and many others think, then there would be no reason to speak of the abstract number as being opposite and having no reference to *jadhr* or *mâl*.[2]

But the responsibility for the origin of this error must be ascribed, to a certain degree, to al-Khowârizmî, himself, and the inaccuracy of his terminology. The same term *jadhr* is very often used to mean the root of a number and as an equivalent of our modern symbol $\sqrt{}$. So, for instance, throughout the chapter "On addition and subtraction"[3] the term *jadhr* stands only for the root. The sentence: "Know that the root (*jadhr*) of two hundred minus ten, added to twenty minus the root of two hundred, is just ten," cannot be rendered otherwise than, as with Rosen,[4] by $20-\sqrt{200}+(\sqrt{200}-10)$. In the same meaning the term *jadhr* occurs in numerous other passages throughout the entire book.[5] The means of determining in what sense the term *jadhr* is used lies in the very nature of the two different ideas. The term *jadhr* meaning the root ($\sqrt{\ }$) has only some sense when applied in the *status constructus* (construct state); *i.e.*, in close connection and interdependence with another word, indicating the number whose root it is; as, for instance, "The root of 200, the root of the *mâl*, its root" etc.; but it can never stand in the *status absolutus* (absolute state), *i.e.*, alone by itself. This is the only way in which al-Khowârizmî distinguishes between the two fundamental meanings of the term *jadhr*. Standing alone it is the same as the *shay'* and denotes the unknown quantity of the first power (x).[6] Standing in the *status constructus*[7] it means the root of the following number.

So for instance is al-Khowârizmî's text of the equation,[8] dealt with by Ruska,[9] both puzzling and confusing. By applying the above mentioned test, however, it becomes clear. In the sentence *jadhr-mâl fa-jadhr-mâl illâ jadhr*

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[1] Loc. cit., p. 64. Instead of squares he suggests "Zahlgrösse" for *mâl*.

[2] As a matter of fact, however, Ruska is not the only one to represent this opinion. It may suffice here to mention, as one instance, the medieval Hebrew mathematician Ibn Ezra (*c.* 1150). In the introduction to the chapter on arithmetical roots (*Sefer ha-Mispar*, ed. Silberberg, Hebrew text, p. 61; German text, p. 65; note 137 on p. 112) Ibn Ezra says: "All the numbers can be divided into three classes: the one (class) are roots, the second are squares, and the third are neither roots nor squares." Evidently this classification was taken from the algebra of al-Khowârizmî, but erroneously applied to pure arithmetic, where it is obviously meaningless and absurd.

[3] Arabic text, pp. 19–21; English text, pp. 27–31.

[4] Ibid., p. 27.

[5] P. 4, lines 2, 5, 6, 11, 14; p. 5, lines 10, 14, 15; p. 6, lines 8, 10, 18, 19; p. 7, lines 9–10, etc.

[6] Arabic text, p. 3, lines 10, 11, 13, 14, 16, 17; p. 4, lines 1, 4, 6, 8, 18, 19; p. 5, lines 1–3, 5, 7, 9, 11, 12, 15, 17, 19; p. 15, lines 4–5; p. 16, lines 15–16, etc.

[7] See note 5.

[8] Arabic text, p. 47 line 12 et seq.; English text, p. 66, line 17 et seq.

[9] Loc. cit., p. 65.

the first two words *jadhr* are in the *status constructus*[1] and denote the root $\surd$, while the third one stands in the *status absolutus* and means the first power (x). In transforming these words into algebraic symbols we shall have to set up the equation $\surd x^2+\surd(x^2-x)=2$ in the form $x+\surd(x^2-x)=2$ as Rosen did and not $\surd x+\surd(x-\surd x)=2$ as Ruska suggested.

It is true that, according to Ruska's theory, there is "eine vollständige Verwirrung des Textes,"[2] a complete confusion in the text of the two chapters "On Multiplication" and "On Addition and Subtraction";[3] for the chapter "On Multiplication" contains also the operations of addition and subtraction, while in the chapter "On Addition and Subtraction" the operations of multiplication and division are also given. There then follows a new chapter "On Division"[4] where we find also operations of multiplication. It seems, however, that the arrangement of the text is clear and justified. The confusion took place only in the headings of the chapters which were added by some ignorant scribe. The difference between the two main chapters does not consist in the difference between multiplication on the one side, and addition and subtraction on the other side, but it consists in the difference between the two kinds of *jadhr*, the one being a *shay'* (concrete number) and the other one being a root.[5]

It is thus clearly established that al-Khowârizmî connects two different ideas with the term *jadhr*, namely, the algebraic first power x and the abstract root of a number. We shall now attempt to explain why al-Khowârizmî introduced a new definition of the term *jadhr*, namely, the conception of the side multiplied into the square unit ($x \cdot 1^2$), discussed in the writer's former article.

This new conception of the *jadhr* is to be found only in the chapter on geometry,[6] and while giving the geometric illustrations for the equations $x^2+21=10x$, and[7] $3x+4=x^2$. It is evident that, while trying geometrically to illustrate these equations, we have to represent x^2 by a square erected upon the side x, the number 21 as 21 square units, and $10x$ as ten rectangles having x for the length and 1 for the breadth [$10(x \cdot 1^2)$]. Otherwise, if 21 would represent only pure numbers, for example, and $10x$ only length-measures, there could be no equation. The same thing, of course, holds good for all quadratic equations in general. All the three members of such an equation must in reality be

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[1] The hyphen between *jadhr* and *mâl* was added by the writer.

[2] Loc. cit., p. 25, note.

[3] Pp. 15–21; English text, pp. 21–31.

[4] Pp. 20–21; English text, pp. 29–31.

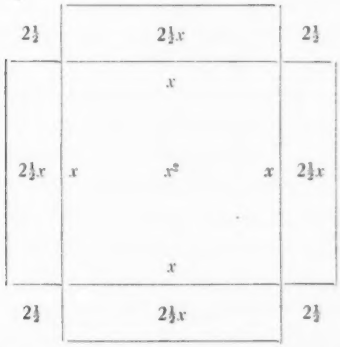
[5] That these headings were not in the original appears from the early Latin translations. Gherard of Cremona (Libri, *Histoire des sciences mathématiques*, vol. 1) has the headings *Capitulum multiplicationis* (p. 265), and *Capitulum aggregationis et diminutionis* (p. 269), but no chapter on division. Robert of Chester (ed. Karpinski) has entirely different titles.

[6] P. 51, lines 4–5; English text, p. 71.

[7] P. 11, line 11; p. 13, line 9 et seq.; English text, pp. 17–19.

conceived as square magnitudes in order to have a real equality, and this necessity is brought to our consciousness by trying to give geometric demonstrations for the equations.

Now it is certainly very strange that al-Khowârizmî does not give the same definition in the two geometric illustrations of the first quadratic equation $x^2+10x=39$.[1] It would seem that this can be explained only in a psychological way, namely, that al-Khowârizmî did not revise his work thoroughly and systematically enough. In the illustration of this first equation he still says that the *jadhr* is the side of the square, and $10x$ are ten sides of the square x^2. We have therefore, says al-Khowârizmî, to divide $10x$ into four parts, each part being $2\frac{1}{2}x$, and to add each part to one side of the original square x. The result is that we have four new rectangles added to each side, consisting of the length x and the breadth $2\frac{1}{2}$. Here the idea must have come into his mind



that if we assume $10x$ to be four rectangles having the length of x and the breadth of $2\frac{1}{2}$, or, as in the second demonstration, to be two rectangles of the length x and the breadth 5, then it can also be ten rectangles with the length x and the breadth 1; or, what is the same thing, the length x multiplied into the square unit. In this way he came to the perception that the *jadhr* cannot be a side, but rather a side multiplied into the square unit, and so he formed a new definition which he used in the two following demonstrations and in the chapter on geometry. But he did not care to rectify the definition given in the first demonstration,[2] that the side of a square is its *jadhr*.

As a consequence of this new geometric definition the square x^2 is not to be resolved into two equal factors $(x \cdot x)$, but into two different factors. The one x represents the *jadhr* $(x \cdot 1^2)$ and the other x represents the pure number a.[3] Or, as it was worded in the writer's previous article: "In order to get the area

[1] Pp. 8–11; English text, pp. 13–16.
[2] P. 8, last line; p. 9, line 1, 15; English text, pp. 13, 14.
[3] Arabic text, p. 13, lines 9–11; English text, p. 19.

of any figure we must first multiply the one side by the square unit; this is then the basis to be multiplied by the other side. We do not multiply one side by one side, but one *jadhr*, representing the square basis, by one side representing the pure number."

It is now interesting to find that this rather strange process of squaring the area is preserved in the Rhind Mathematical Papyrus as the oldest method of computing the area used by the Egyptians as early as 1650 B.C. Thus it is quite possible that in formulating this geometric definition of the *jadhr* al-Khowârizmî was prompted, not only by mathematical reasons, but also by old historical reminiscences of the Egyptian geometry.

The Egyptian unit of length in the measurement of land was the *Khet* of 100 cubits. The commonest unit of area was the *setat* or square *Khet* which contained 10,000 square cubits. For practical purposes in land measuring they used a unit called the cubit-of-land or the cubit-strip. It was a narrow strip of land 100 cubits long and one cubit broad. Smaller portions of a *setat* were expressed in such cubit-strips.[1] To get the area of a rectangle they sometimes multiplied its length in cubits by its width in *Khet*. This gave them the correct answer in cubit strips.[2] Still, more clearly we see in Problem 48 that for the multiplication of 8 *Khet* into 8 *Khet* and of 9 *Khet* into 9 *Khet* "the working actually written looks like the multiplication of 9 *setat* by the pure number 9."[3]

Thus the Egyptian computed the square by first multiplying the one side (9 *Khet*) in the square unit. This was the square-basis (*jadhr*) to be multiplied by the other side, representing the pure number. The same process is also described by al-Khowârizmî[4] who says: "And in every square figure one of its sides multiplied into the square unit is its *jadhr* . . . . and we make the other side, namely, *hj*, three, and this is the number of its *jadhr's*."

This archaistic way of computing the square finds its justification not only in "the peculiar Egyptian system of multiplication"[5] but in the very nature of primitive computation. The ancient Egyptians did not compute areas and measure their fields according to abstract rules. They originally found the area in an experimental practical way by taking a small square measure and trying out how many times it was contained in the field to be measured, "as we today measure cloth by the yard."[6] For this purpose the mere length-

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[1] J. E. Peet, *Rhind Mathematical Papyrus*, London, 1923, p. 24 et seq. A. B. Chace, *Rhind Mathematical Papyrus*, New York, 1927, vols. 1 and 2, examined in proof sheets, chapter on "Measures of Area."

[2] Chace, loc. cit. Chace does not quote expressly the examples but he probably refers to Problem 49.

[3] Peet, loc. cit., p. 89.

[4] P. 13, line 9 et seq.; English text, p. 19; quoted also this Monthly, vol. 33, p. 263, note 4.

[5] Peet, loc. cit.

[6] Chace, loc. cit.

measure, the *Khet* or cord, could not be of any use. They had, therefore, to create the square-unit, the cubit-strip, being a *Khet* long and one cubit broad. This was the first *jadhr*, square-basis, used in practical life.

This cubit-strip might also, in all probability, be the underlying idea for the Egyptian conception of the square root. The idea of the square root existed in Egypt and the technical term for it was *Ḳnbt*, literally "corner" or "angle." Peet[1] has the explanation for that, that the length of each of the two sides of a square containing any corner of it was its square root. Yet it is not the pure, one-dimensional side which contains a "corner" but only the *jadhr*, the side multiplied with the square unit, contains a "corner." "Corner" or "angle" is the primitive word for a small square unit. Apollonius (*c.* 225 B.C.) defines the angle as the contracting of a surface at one point under a broken line.[2] Since the word "corner," "cornerstone" usually implies also the meaning "lowest part," "basis, foundation" we might see in the Egyptian *Ḳnbt* the source and origin of the Greek *pythmenes* (bases), the Arabic *ass*[3] and *jadhr*, the Hebrew *iqqâr* and *ush* and the Hindu *mûla*.[4]

Thus we are enabled to pursue the origin of our term "root" to an algebraical term denoting the concrete number with a basis, the first power (x) and to a geometric term denoting the basic square unit ($x \cdot 1^2$). All the three conceptions are contained in the Arabic *jadhr* and are preserved in the definitions of al-Khowârizmî.[5]

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[1] Loc. cit., p. 20.

[2] See Smith, *History of Mathematics*, vol. 2, p. 277. Compare also the definition of Heron (*c.* 250 A.D.) quoted by Tropfke, vol. 4, 2nd ed., p. 25.

[3] Al-Qalsadi, quoted above, says that the concrete number (*shay'*) has an *ass*; *i.e.*, it contains a basis or "corner."

[4] See the Article of Datta "*On Mûla, the Hindu Term for Root*," in the October issue of this Monthly (vol. 34, pp. 420–423). The writer is gratified to see that his paper inspired the distinguished Hindu scholar Datta to cooperate on the same subject and to give us an authoritative account of what the Hindu sources say about the square root. Since the term *mûla* is found already in the works of Āryabhaṭa (499) the writer will modify his statement and say that the Arabic *jadhr* is possibly due to the influence of the Hindu *mûla*. This is also the opinion of Woepcke (*La propagation des chiffres Indiennes*, p. 68) and Ruska (loc. cit., p. 68). But in consideration of the fact that al-Khowârizmî was not acquainted with the methods of Hindu algebra (see Leon M. Rodet, *L'Algèbre d'Al-Khârizmî*, Extrait du Journal Asiatique, Paris 1878) we may also refer to the Hebrew *iqqâr*. The latter term occurs in the oldest Hebrew geometry, *Mishnath ha-Middoth*, which, according to Shapiro's and the writer's opinion, was written *c.* 200 of the Christian era, and which was almost verbally copied by al-Khowârizmî. On the other hand the writer's opinion is supported by Datta, who informs us that Āryabhaṭa had in view also the concrete concept of the side for the term *mûla*, and that the terms *mûla* and *pada* imply also the meaning of "basis, foundation." Furthermore, the term *pada*, "part," "portion," "square on chessboard," throws more light on the writer's theory concerning the origin of the terms *takstr*, *thishboreth*, for area; see this Monthly, vol. 34, p. 80 et seq.

[5] The writer wishes to express his thanks to Professor David Eugene Smith for the opportunity of examining the manuscript of Dr. Datta's article and the proof-sheets of Dr. Chace's edition of the Rhind Papyrus before publication, and for his critical help in the final preparation of this article.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. NOTE ON THE QUATERNARY QUADRATIC IDENTITY OF HERMITE.

By E. T. BELL, California Institute of Technology

In a recent number of this Monthly,[1] C. G. Latimer showed that Hermite's important identity is implied by its apparently very special case, namely Euler's theorem on four squares. The form concerned is

$$x_0^2 + \sum_{i,j=1}^{3} A_{ij}x_ix_j, \tag{1}$$

where A_{ij} is the cofactor of a_{ij} in the determinant $|a_{ij}|$, $a_{ij}=a_{ji}$. Hermite's identity exhibits the repetition of this form under multiplication. In previous papers[2] I gave a method for generalizing such multiplication theorems by the introduction, through irrational transformations of the variables, of parameters into the coefficients of the given forms. If in any particular case this method leads to a form which could have been obtained from the original by a rational integral transformation of the variables, with rational integral coefficients, it is clear that the final multiplication theorem is no more general than the original. For example, if $f(x, y)$ repeats under multiplication, we learn nothing new if we replace (x, y) by (ax, by), where a, b are rational. That the irrational transformations mentioned do not always lead to a mere restatement of the original theorem, is shown by the generalization of the eight square and other identities in the papers cited. To prove, therefore, in any case that such generalization by irrational substitution is impossible is not trivial.

A straightforward application of the method mentioned, precisely as in the generalization of the foursquare theorem, readily shows that Hermite's form (1) does not admit of generalization by the introduction of parameters into the coefficients.

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[1] Vol. 34 (1927), pp. 363–4.

[2] Annals of Mathematics, vol. 27 (1925), pp. 99–104; Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 71–80.

II. Note on Stirling's Numbers

Jekuthiel Ginsburg, Teachers College, Columbia University

In Mr. Bennett's beautiful solution of problem 3161 (in the February, 1927, issue of this Monthly, p. 100), there appears a set of numbers given by the relationship

$$_{i+1}U_j = (j+1)_iU_j + {}_iU_{j-1}.$$

From these numbers, the writer adds, a triangular array may be formed which is similar to Pasçal's triangle.

It is to be regretted that the author did not tabulate the values of the U's in the triangular form referred to. The beauty of the solution would have been shown to greater advantage and at least some of the readers would have identified the U's as that extremely interesting and very important set of numbers which have attracted the attention of some of the best mathematical minds for the last two centuries[1] and which have been named by Professor Nielsen of the University of Copenhagen "The Stirling Numbers," in honor of the man who first discovered them and conceived their importance.

This identification greatly simplifies the task of obtaining solutions of the problem referred to, since tables of the Stirling Numbers have been compiled by various writers.

We would have then

$$\phi_0 = e^x; \quad \phi_1 = \frac{dx\phi_0}{dx} = (x\phi_0)' = e^x(x+1); \quad \phi_2 = (x\phi_1)' = e^x(x^2+3x+1);$$

$$\phi_3 = (x\phi_2)' = e^x(x^3+6x^2+7x+1); \quad \phi_4 = (x\phi_3)' = e^x(x^4+10x^3+25x^2+15x+1);$$

and the triangular array would then be of the form

1
1, 1
1, 3, 1
1, 6, 7, 1
1, 10, 25, 15, 1
1, 15, 65, 90, 31, 1

The Stirling Numbers are characterized by many very beautiful properties and have an interesting history. They have been discovered and rediscovered, cropping out again and again under various disguises, in almost every branch of mathematics. Even some of those who considered them so important as to compile and publish extended tables did not follow up their ramifications in branches outside of the one in which they were particularly interested.

Stirling[2] devised them for use as a tool by means of which he could write

[1] The list includes, among others, the names of Euler, Vandermonde, Cauchy, Sylvester, Cayley, Crelle, Grunert, Boole, Schlömilch, Nielsen.

[2] *Methodus differentialis*, London (1730), p. 6.

any given polynomial in the form of a series of factorials,—an object which was considered very important until Weierstrass proved its futility.[1]

A mathematician addicted to factorials would write

$$x^2 = x(x-1) + x,$$
$$x^3 = x(x-1)(x-2) + 3x(x-1) + x,$$
$$x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x, \quad \text{etc.},$$

the coefficients being the original Stirling Numbers, which may be seen to be identical with those in Mr. Bennett's solution.

The next appearance of the Stirling numbers was in the theory of finite differences, as submultiples of the differences of zero. But neither Boole,[2] who used De Morgan's tables, nor Cayley,[3] who computed extended tables of his own, realized that these numbers are certain multiples of the numbers which Stirling was using for his purposes. (The proof of their identity is too extensive to be given here, but the fact may easily be verified.)

The theory of finite differences has a kindred relationship to the practice of interpolation, and this is the reason for the attention bestowed on the Stirling numbers by writers of textbooks on interpolation and by astronomical observers. A very extended table of Stirling numbers was compiled by the astronomer, N. P. Bertelsen and was published in Thiele's *Interpolationsrechnung*,[4] although the most extended table is probably Glaisher's.[5]

Another event in the history of the Stirling numbers is seen in their appearance[6] as factors of coefficients of the expansion of $(e^x-1)^n$ in terms of x and as coefficients in the formulas expressing the sum of equal powers of the series of natural numbers.

For example, Professor I. J. Schwatt has, in his *Introduction to Operation with Series* (Philadelphia, 1924), the following formulas (p. 88):

$$\sum_1^n k^2 = \binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3},$$
$$\sum_1^n k^3 = \binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4},$$
$$\sum_1^n k^4 = \binom{n}{1} + 15\binom{n}{2} + 50\binom{n}{3} + 60\binom{n}{4} + 24\binom{n}{5},$$

[1] Crelle's Journal, vol. 51 (1860).
[2] *Finite Differences*, p. 20. Cf. De Morgan, *Differential Calculus*, p. 253.
[3] Transactions of the Cambridge Philosophical Society, 13 (1873), pp. 1–5.
[4] Leipzig, (1909), pp. 31–2.
[5] Quarterly Journal of Pure and Applied Mathematics, vol. 31 (1900), pp. 27–28.
[6] Cauchy, *Exercices*, III (1828), p. 346; Schlömilch, Crelle's Journal, vol. 44, p. 344.

$$\sum_1^n k^5 = \binom{n}{1} + 31\binom{n}{2} + 180\binom{n}{3} + 390\binom{n}{4} + 360\binom{n}{5} + 120\binom{n}{6}.$$

A comparison with the coefficients in the table previously given will help identify the above as the Stirling numbers, multiplied by the corresponding factorials (2!, 3!, etc.)

But $\sum k^p$ plays a very important part in the study of Bernoulli numbers. Hence the intimate connection between the numbers of Bernoulli and those of Stirling. In fact there is a way to show that from at least one point of view the Bernoulli numbers may be regarded as a species of the Stirling numbers.

The interesting career of the Stirling numbers is too long to be compressed in a note of the kind presented here. It will suffice to call attention to what seems to be the most convenient mode of approach. Accepting Professor Nielsen's division of the Stirling numbers into two classes,[1] the Stirling numbers of the first class corresponding to a number n may be defined as the sums, S, of the products of the first n integers taken as factors $1, 2, 3, \cdots, p$ at a time without repetitions, while the Stirling numbers of the second kind corresponding to a number n are the sums, S', of the products of the first n integers taken as factors $1, 2, 3, \cdots, p$ at a time with repetitions. With the help of these definitions the Stirling numbers may be derived without recourse to the numerous functions, in the development of which they play such an important part.

It may also be of interest to note, that, for any n, the sum S of the products of the first n integers taken two at a time may be directly obtained by performing the algebraic division of the numerator of $(2+x)/(1-x)^5$ by the expanded denominator. A similar fraction for the sums of products S', with repetitions, is $(1+2x)/(1-x)^5$. In fact, by actual division,

$$(2 + x)/(1 - x)^5 = 2 + 11x + 35x^2 + 85x^3 + \cdots$$

and

$$(1 + 2x)/(1 - x)^5 = 1 + 7x + 25x^2 + 65x^3 + 140x^4 + \cdots.$$

Similarly for the case of the factors taken 3 at a time, the Stirling numbers of the first kind will be expressed by the coefficients of the expansion of

$$(6 + 8x + x^2)/(1 - x)^7 = 6 + 50x + 225x^2 + \cdots,$$

while the Stirling numbers of the second kind will be expressed by the coefficients of

$$(1 + 8x + 6x^2)/(1 - x)^7 = 1 + 15x + 90x^2 + 350x^3 + \cdots.$$

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[1] *Theorie der Gamma Funktion* (1904), p. 65.

Similarly the values of ${}_nS_4$ and ${}_nS_4'$ will be given by the coefficients of the expansions of

$$(24 + 58x + 22x^2 + x^3)/(1 - x)^9 \text{ and } (1 + 22x + 55x^2 + 24x^3)/(1 - x)^9$$

respectively.

The above relations give the following formulas for the Stirling numbers or, what is the same, for the corresponding combinatorial products of the first n integers taken 2, 3, and 4 at a time without or with repetitions.

$${}_nS_2 = 2\binom{n+2}{4} + \binom{n+1}{4}$$

$${}_nS_3 = 6\binom{n+4}{6} + 8\binom{n+3}{6} + \binom{n+2}{6}$$

$${}_nS_4 = 24\binom{n+6}{8} + 58\binom{n+5}{8} + 22\binom{n+4}{8} + \binom{n+3}{8}$$

$${}_nS_2' = \binom{n+2}{4} + 2\binom{n+1}{4}$$

$${}_nS_3' = \binom{n+4}{6} + 8\binom{n+3}{6} + 6\binom{n+2}{6}$$

$${}_nS_4' = \binom{n+6}{8} + 22\binom{n+5}{8} + 58\binom{n+4}{8} + 24\binom{n+3}{8}$$

The proof of the above formulas, which the writer believes to be new, and the extension to sums of higher order are beyond the scope of this article.

III. Beginning Geometry and College Entrance

By Ralph Beatley, Harvard Graduate School of Education

In 1923 the College Entrance Examination Board published its Document 108 embodying a detailed statement of the revised requirements in plane and solid geometry. Those who were charged with the preparation of this document gave heed also to a suggestion of the National Committee on Mathematical Requirements[1] with respect to the desirability of introducing the more elementary notions of solid geometry in connection with related ideas of plane geometry, and prepared accordingly the syllabus for Geometry *cd*, the so-called Minor Requirement in Plane and Solid Geometry. Document 108 states that "this

[1] See p. 37 of their report published in 1923 under the auspices of the Mathematical Association of America, Inc.

requirement is designed to cover the most important parts of plane and solid geometry, in such a way that the preparation for it can be completed in the time usually devoted to the standard requirement in plane geometry," *i.e.*, one year.

It is possible that those who prepared this syllabus did not gauge with sufficient accuracy the amount of material which it was proper to include in a one-year course of demonstrative geometry. If that is so, it should not be difficult to make a suitable adjustment when more evidence is at hand on which to base a judgment. The requirement should probably stand as it is for the time being. Certain schools are interested to give it a fair trial, adequate texts are available, and the colleges are not unsympathetic.

It is common knowledge that solid geometry is less widely taught than formerly. That is, a smaller proportion of students completing plane geometry go on to solid geometry than heretofore. This may simply be the result of the students' preference, freely expressed under elective systems of one sort or another. Even a teacher of mathematics, while not discounting the value of the three dimensional experience, might well question the desirability of devoting a year and a half to demonstrative geometry in the secondary school. No one knows the relative value of eight months of geometry as compared with five; nor is there any information available as to just how many months the student should demonstrate and argue about geometric abstractions before he attains in reasonable measure to those disciplinary and cultural aims which we like to think our teaching of geometry achieves. There does seem to be widespread agreement, however, that the student should not bring his geometric experience to a close without having considered the three dimensional aspects of the world about him. Even before the recent decline in the popularity of solid geometry, our instruction in the mathematics of three dimensions—in colleges as well as in secondary schools—suffered in comparison with that of European schools. We do not need to remain forever complacent about this unfavorable comparison, simply because it is now so familiar to us.

So long, however, as the requirements of the College Entrance Examination Board permit of a choice between Plane Geometry (*C*) and Plane and Solid Geometry, Minor Requirement (*cd*), and so long as human nature remains human, teachers of geometry in secondary schools will tend to continue the same old round and will be slow to follow the lead of those more adventurous spirits who are trying to give in the first—and, for most students, the only—year of geometry not only an adequate training in close reasoning and argumentation, but also an appreciation of the geometry of three dimensions. The number of candidates presenting themselves each year for examination in the minor requirement in plane and solid geometry (*cd*) is pitifully small, and there is good reason to believe that most of these take the examination by

mistake. It is not out of place to suggest that the proposal to include a mite of trigonometry, and another mite of logarithms, in the revised requirement of elementary algebra would be today in equally low repute if the new requirement were merely an optional substitute for the old.

Should then this new minor requirement in plane and solid geometry be made the sole requirement, and should the old examination in plane geometry be discontinued? No: that would not be fair. This new requirement in geometry differs from the old far more than was the case in algebra. Furthermore, it is not certain that it is not a bit too long.

Should the new requirement in geometry be withdrawn because so few seek to prepare for it? No: that too would not be fair. The schools have not had time to give it a fair trial. Even in the case of the trigonometry and logarithms in the algebra requirement, the schools—though meeting the requirement fairly well by now—can hardly be said to have had sufficient time to give a thorough appraisal of the new material and the results of teaching it. No: we need this new requirement constantly before us as an ideal to measure up to. It may possibly prove desirable to modify it very slightly. But to modify it so drastically as to make it tomorrow a fair requirement for all schools, and no longer optional, would cut the heart out of it.

Why not rather modify requirement *C*, making changes gradually over a period of years, and replacing items in plane geometry by items in solid geometry no faster than the traffic can stand? For a beginning, why not expect the candidate to be familiar with simple mensurational work in three dimensions, so that he could meet with confidence a question on the examination paper involving the total area of a house, or cylinder, or silo? Such a question would be necessarily one of the easy originals on the paper. This slight addition to the requirement could be easily balanced by the omission of a few unstarred theorems, or by some other equitable adjustment. In any case this proposal is not intended to make the present requirement easier. The effect of this change would be to shift very slightly the emphasis from 'plane geometry only' to 'plane and solid geometry.' The present ratio of book theorems to originals could—and probably should—stand. At any rate, the emphasis on originals should not be reduced.

This is admittedly only a feeble beginning. But even this little can be very significant. The mensurational work mentioned above has commonly been taught under the head of arithmetic in the eighth grade. It is now appropriately included in the geometry of the junior high school. As such its status is similar to that of the numerical trigonometry now included in the algebra requirement, or of the spelling and grammar implicit in the requirement in English.

Later it ought to be possible to add to the requirement certain book theorems from solid geometry: only a few, if they are to be starred theorems; a dozen or more, if unstarred. In the latter case one of these unstarred theorems could

appear in the role of an original on the examination paper, just as can now occur in the case of unstarred theorems in plane geometry. However the change is effected, there should of course be a suitable reduction in the plane geometry part of the requirement.

The experience of the past few years with the requirements as they now stand in Document 108 has indicated the desirability of a few minor changes in the starring of theorems, and the like. When such changes are made it would be a simple matter to incorporate in the present requirement *C* (Plane Geometry) some slight modification in keeping with the suggestion made above. The title of the requirement might then appropriately be changed to "Demonstrative Geometry."

There will always be room for improvements in this and other syllabi also. As these periodic revisions are made, the change in emphasis in Requirement *C* toward the goal now set for us in Requirement *cd* can be gradually and painlessly accomplished. "In view of the changes taking place at the present time in mathematical courses in secondary schools, and the fact that college entrance requirements should so soon as possible reflect desirable changes and assist in their adoption, the National Committee recommends[1] that either the American Mathematical Society or the Mathematical Association of America (or both) maintain a permanent committee on college entrance requirements in mathematics, such a committee to work in close cooperation with other agencies which are now or may in the future be concerned in a responsible way with the relations between colleges and secondary schools."

At present there is little to encourage the schools to make the change in their teaching of geometry which has been recommended above. It is practically necessary that any such change be accompanied by a simultaneous change in college entrance requirements if it is to be generally effective. This recommendation of a gradual modification of Requirement *C* to approach in character Requirement *cd* has for its main purpose, therefore, the improvement of instruction in geometry in our secondary schools. This is the important thing, and the question of college entrance is raised merely as a means to that end. It would be a shame were we teachers of geometry to witness the gradual disappearance of the extra half year's work in three dimensional geometry from the curriculum without doing anything to save some remnant for the bulk of our students, who seem destined from now on to have but one year to devote to demonstrative geometry.

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[1] On p. 48 of its report.

IV. On Zeros and Pseudo Zeros

By Luise Lange, Chicago, Illinois

The following discussion is meant to draw attention to a well-known though rather neglected point of mathematical practice, notably school practice. The name under which this point usually goes is that of "the consideration (or rather lack of consideration) of the limits of accuracy." The neglect of this point frequently leads to the calculation of results to a fictitious degree of accuracy, that is to an accuracy greater than is warranted by the data used in the calculation. Yet in thinking over what is really at the bottom of this pseudo-accuracy we became aware that the whole issue turns on one very simple fact which we wish to call *abuses of the zero*. School mathematics, from the grades to the colleges, has made it a habit to abuse the zero. This contention we shall now endeavor to establish in the hope of bringing about a discussion on the question whether it may not be worth the effort—which it would doubtless cost—to rid school mathematics of this habit.

To state our case we first have to say a few words about the difference in the meaning of the term "number" according to whether one has to do with abstract or with denominate numbers. Granting the abstract integer numbers, each element of the continuous set of abstract numbers (we may limit ourselves here to real numbers) is defined by certain algebraic or transcendental operations on the integers. These operations may be regarded as rules to compute the particular number exact to *any* number of decimal places. In the case of rational numbers every further place is known, once its period has been found; for irrational numbers each place has to be computed individually. But, and this is the point, every place is defined by the one rule and can be found if desired.

Denominate numbers, on the contrary, are defined by concrete operations on given physical objects. If this operation is that of *counting* discrete, finite sets of things, a one-one correspondence exists between the so defined denominate numbers and the abstract integers. If, however, the operation is that of *measuring* a continuous magnitude, the so defined denominate number corresponds, *not to any one definite abstract number, but to a whole range in the continuous set of real abstract numbers*. For the "definition" of a denominate number extends necessarily only to a finite number of places, for the following reasons:

(1). Physical magnitudes cannot be *defined* beyond a certain limit of accuracy. They all are reducible to the measurement of three kinds of quantities, length, time, and mass. Now a length is undefined inasmuch as its extremes are not mathematical points. The time of an event is undefined inasmuch as every event occupies a duration, no matter how small. And unless the exact

moment (watch the limitation on this notion) of the measurement be given, a length or a mass is undefined inasmuch as it is subject to minute variations in time. Thus to give the distance between a ship and a light house exact to 10^{-4} inches, the earth's radius exact to yards, the time of a flash of lightning to 10^{-6} seconds, the weight of a certain piece of iron to 10^{-10} grams, would not only be practically difficult, but *devoid* of meaning.

(2) Even within the scope in which the magnitude can be defined the subjective and objective limitations in the accuracy of measuring reduces every denominate number to an essentially limited number of places.

(3) In the majority of cases denominate numbers are given only to a *voluntarily limited number of places*, exact "to the nearest" inch, gram, etc. The reasons for this may be twofold: (a) there may not be any interest in knowing the magnitude more accurately for the particular purpose; (b) if the number is wanted, not for its own sake, but only in connection with others for the computation of some other magnitude, then the relative accuracy of the various measurements should be adjusted in realization of the fact that "a chain is no stronger than its weakest link."

Any denominate number then, being intrinsically determined and defined only to a limited number of places, corresponds, as said before, to a whole range in the continuous set of abstract numbers. All abstract numbers are equally well suited to represent that denominate number if they agree with it in the known places though differing from each other in higher places. The width of the range depends on the accuracy of the measurement; in particular, if the measurement is "accurate to the nearest unit u," the width of the range is in general $\pm .5u$, unless specific error limits are given.

Now it is by not keeping in mind this essential peculiarity of denominate numbers that what we called the abuse of the zero occurs. First in its milder form it is this: In writing denominate numbers the zero symbol is being used in a twofold rôle. Like the other digits it indicates the number of units in a place saying that they are nought (as when we write 1001 nights); but unlike the other digits it is also written instead of unknown (because non-defined or non-measured) places in order to impart the proper place value to those which are known. For example, in giving the velocity of light as 300,000 km/sec, not all of these zeros stand for true noughts; some of the last ones are mere "dummies" put in the place of unknown numbers to assure the proper place value to the 3. This ambiguity of the zero symbol would be avoided if one would generally adopt the practice of the "exact scientist" and express the place value, not by means of dummy zeros, but by powers of ten, writing digits only in those places for which they are actually known. For example, $c = 3.0 \times 10^5$ km/sec would mean that the value of c is known between the

limits $2.95\times10^5<c<3.05\times10^5$, etc. Yet, according to the practice of many, this form would be as ambiguous as the other because of a second and worse form of the habit of abusing the zero, namely the habit of dropping or adding zeros at will behind the last given decimal place. For example, the statements that a certain distance is 15 feet, 15.0 feet, 15.000 feet are treated as equivalent and interchangeable. Most people, it is true, may never have written 15.000 feet for 15 feet, though surely many have written 15 feet instead of 15.000 feet; but nevertheless they have *treated* the 15 as a 15.000 if they ever gave as its logarithm the value 1.17609 instead $1.17\pm.01$. The fact however is that if a place is not given it is *unknown*, not *zero*. If it were known to be zero it should be given as such. To put a zero into an unknown place really amounts to a falsehood, to the pretension of knowing more than one does know. Moreover, when the zeros are continued to places to which the magnitude cannot even be *defined*, they become downright nonsensical.

To illustrate the above we shall next give a few samples of typical problems in which, according to the aforementioned habit, zeros are being abused so that results of a fictitious accuracy are obtained. We shall indicate by the symbol x each instance where in thought or in fact a zero is usually substituted for an unknown number; by bold face type, a numerical value obtained by operations involving pseudo-zeros; also by a dot under a number we indicate that its value is uncertain due to insufficient information on the following places.

(1). A school room is 32 feet long, 26 feet wide, and 12 feet high. How many cubic feet does it contain?

32	832
26	12
192	1664
$64x$	832
832	9984

The answer is $v=9.9\times10^3$ cubic feet $\pm$ several hundred cubic feet. The pseudo-exact answer would be $v=9984$ cubic feet.

(2) The same problem computed by means of logarithms.

$\log 32.xxx = 1.50515$
$\log 26.xxx = 1.41497$
$\log 12.xxx = 1.07918$

$\log v = 3.99930$

$v=9984.0$ cubic feet.

(3) A lamp post is 9.0 feet high; it is casting a shadow 5.2 feet long. Find the angle of elevation of the sun at this moment.

$$\tan e = \frac{9.0}{5.2}$$

$\log 9.0xxx = .95424$
$\log 5.2xxx = .71600$
$\log \tan e = .23824$

The pseudo-exact answer is $59° 58' 54''$.

$e = 60° \pm$ about $1°$.

This symbol x for the pseudo-zero, I believe, makes it quite obvious how the error in the original measurement affects the result. At the same time it also serves well to demonstrate to what extent school mathematics is dealing with pseudo-zeros and with pseudo-accurate results. And it is not only the unaware teacher who accepts or even urges these unreasonably numerous decimal places; the textbooks, fairly alike for all types of schools, encourage this habit, (a) by disregard for the relative accuracy of the given data in a problem, (b) by the pseudo-accurate anwers printed in the back of the book. This contention hardly needs any demonstration, however I might mention that the above example (3) was taken from one of the most widely used college text books, only with this correction: in the book the post is given as 9 feet not 9.0 feet high, its shadow 5.2 feet. Though this would impair the result still further, it is nevertheless given exact to seconds, that is a result exact to $3 \cdot 10^{-3}\%$ is found on the basis of measurements exact to only about 10%.

We further wish to make this point: Aside from being mathematically objectionable, this failure to treat the limited number of places in a denominate number as an essential feature is, in our opinion, also regrettable from the pedagogical point of view. (1) One thereby misses a valuable chance to make the student use his own judgement; for questions like "May we use the slide rule?" or "Do you want us to interpolate?" would be self-prohibitive if the students were trained to regard the required accuracy in a problem not as something extraneous, to be imposed by the authority of the teacher, but as determined by the given data and the purpose. (2) If one would persistently emphasize this point, one would incidentally develop the much sought-after "functional thinking"; for the student would continually realize that the errors in the measurement affect the result, and to what degree. (3) The faulty habits acquired in high school and college training are most likely to be carried over into professional practice where they are objectionable not only from a theoretical but from an extremely practical point of view; for there they lead to such concrete evils as false information, and wasted time and money. This is a contention which anybody who ever worked as an "exact scientist" in an industry can illustrate from his own experience.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

NEW BOOKS RECEIVED

BROAD, C. D. *Sir Isaac Newton.* (Annual lecture on a master mind, Henriette Hertz trust of the British Academy.) London, Oxford University Press, 1927. 32 pages. $0.70.

DAVIS, H. T. *A survey of Methods for the Inversion of Integrals of Volterra Type.* (Indiana University Studies, Numbers 76, 77.) Bloomington, Indiana University, 1927. 72 pages. $1.00.

RAMANUJAN, SRINIVASA. *Collected Papers.* Edited by G. H. Hardy, P. V. Seshu Aiyar, and B. M. Wilson. Cambridge, University Press, 1927. xxxvi+355 pages.

REVIEWS

Mathematical and Physical Papers, **1903–13.** By BENJAMIN OSGOOD PEIRCE. Cambridge, Mass., Harvard University Press, 1926. viii+444 pages. Price $5.00.

This volume contains a collection of twenty one papers which were published by Professor Peirce during the last ten years of his life; it will do much to give him a lasting reputation for mathematical scholarship. The collection does not represent the whole of Peirce's scientific work, for the bibliography at the end of the book gives a list of two books, published by Ginn & Co., and fifty six papers. The papers selected for republication represent Peirce's best work and together form a treatise on an important branch of electricity and magnetism which is inadequately treated in many textbooks. The original papers are inaccessible to many workers in this field and this volume may be comparable in usefulness with a volume of Heaviside's electrical papers. The eighth paper in the book might very well be read first. It is the great memoir entitled "On the determination of the magnetic behavior of the finely divided core of an electromagnet while a steady current is being established in the exciting coil." It commences with a useful historical account of the development of the ordinary mathematical theory of two neighboring circuits and then passes on to the consideration of circuits which contain iron. Some problems relating to eddy currents are then solved with the aid of linear partial differential equations and Bessel functions. Peirce shows in this paper that he has made himself a master of the methods of dealing with the partial differential equations of mathematical physics.

Another paper which is of particular interest to the mathematician is the nineteenth, which is entitled "The effects of sudden changes in the inductances of electric circuits as illustrative of the absence of magnetic lag

and of the von Waltenhoven phenomenon in finely divided cores. Certain mechanical analogies of the electrical problems." The mechanical analogies described in this paper are quite pleasing. One, of Peirce's own invention, is made from three weighted roller skates and some cheap pulleys. This paper contains many interesting diagrams and formulas and, like the other, ends with an elaborate mathematical investigation involving Bessel functions.

The reader will perhaps be anxious to know what is meant by "the von Waltenhoven phenomenon." The answer is given very clearly at the beginning of the next paper from which we take the following quotation:

"If an increasing current I, ending in the maximum value I', be sent through a long solenoid, the final value of the magnetic moment of a solid bar of originally demagnetized soft iron or steel within the solenoid frequently depends, not only upon the strength of the current, but also upon the manner of growth of the current in attaining this intensity. The moment will be greater if the current be suddenly applied in full strength than if it be made to grow slowly, either continuously or by short steps. If, after the current has remained steady for a short time, at the strength I', it be made to decrease to zero, the residual moment of the bar will usually be less if the circuit be suddenly opened than if the decrease be made slowly, by gradually introducing more and more resistance, and the demagnetizing factor of a given cylinder is considerably less when computed from observations made by the method of sudden reversals than if it is determined by the slow step-by-step changes in the exciting current."

This phenomenon greatly interested Peirce and he made a series of experiments to determine the conditions under which anomalous magnetization appears and discussed the matter later theoretically.

The papers do not deal wholly with the growth of currents. There are some papers devoted to the design, use, and improvement of physical apparatus. For instance, the eleventh paper of the set discusses the theory of the ballistic galvanometer of long period, while the first paper is devoted to the design of chilled and seasoned cast-iron magnets for the production of the permanent fields of d'Arsonval galvanometers. The second, third, and fourteenth papers belong to the subject of vector analysis and are of interest to geometers. In the last of these papers Peirce introduces the idea of the derivative of one scalar point function with respect to another.

The book is beautifully printed and illustrated and has as its frontispiece a picture of the author.

H. BATEMAN

Einführung in die Axiomatik der Algebra. By H. BECK. Berlin and Leipzig, Walter de Gruyter & Co., 1926. x+197 pages.

In this book the author approaches the subject-matter of ordinary algebra through a postulational technique. His procedure may be described as follows. He puts down at the outset certain expressions, which have a formal resemblance to certain propositions of ordinary algebra, and deduces, from the expressions put down, other expressions, which have a formal resemblance to other propositions of ordinary algebra. The justification of this procedure lies in the fact that the propositions of ordinary algebra which resemble his derived expressions are logical consequences of those propositions which resemble his original expressions, just as the derived expressions are logical consequences of the original expressions. These expressions, original and derived, are what would commonly be called 'propositional functions,' and the propositions of algebra are values or instances of these propositional functions. This is, of course, the usual postulational analysis with which we are familiar in the writings of Peano, Huntington, and others; and I think that one who is acquainted with these writings will find nothing in this book which is essentially new to him. A number of sets of postulates are given and in connection with these sets the author discusses questions of independence and consistency in the usual way.

I wish to consider one very general feature of the author's method as this method is applied by him to his particular subject-matter. We are much more familiar with axiomatic analysis as applied to geometry than we are with such analysis as applied to algebra; and I think that we are likely to feel that an axiomatic method is less appropriate to the subject-matter of algebra than it is to that of geometry. On the other hand, the author seems to me to hold the view that his procedure is precisely analogous to that of Peano, Veblen, and others, who have developed abstract geometry axiomatically. I think that this view is quite certainly mistaken; and whether the author holds it or not, it is a view which seems to be sometimes held. For this reason I wish to explain as clearly as I can what appears to distinguish the author's analysis of his subject-matter from similar analyses in the case of abstract geometry. I do not wish to maintain that a given species of algebra could not be determined postulationally in precisely the way in which species of abstract geometry are determined postulationally, but simply that the author's postulates do not do this.

Let us consider, in the first place, the following axiom, which is one of the axioms used by Veblen in his analysis of abstract euclidean geometry: "If points a, b, c are in the order abc, they are distinct." It is clear that this axiom, as it is commonly understood, is a propositional function, neither true nor false. But when the variables which occur in it are given appropriate meanings in terms of physical space, it becomes a true proposition. However, this derived proposition, although true, is not a truth of mathematics; and mathematical analysis is not in the least concerned with the fact that the proposition is true.

It follows that a complete mathematical treatment of abstract geometry is a complete mathematical treatment of geometry, abstract or otherwise. It must not, of course, be supposed that in abstract geometry we are not concerned with the truth of propositions at all; there is no intellectual enterprise whatever that is not vitally concerned with the truth or falsity of certain propositions. Let p_1 denote the axiom just given, and let p_2 denote some theorem which follows from this axiom. Then the proposition "p_1 necessitates p_2" is a truth of geometry. Now consider the algebraic axiom, $a+(b+c)=(a+b)+c$, which occurs among the author's axioms of addition. Taken abstractly, this axiom must be held to be a propositional function, neither true nor false; but when appropriate values are assigned to the variables, it becomes a true algebraic proposition. Now this resulting proposition is a proposition with the truth of which mathematics is vitally concerned; and this is precisely what is not the case in the corresponding illustration taken from geometry. In order to carry out the analogy, we may speak of "abstract" algebra and "concrete" algebra. It is clear that a complete mathematical treatment of "abstract" algebra is not a complete mathematical treatment of algebra, "abstract" and otherwise. The fact is that the analogy breaks down in important respects. Ordinary algebra is already abstract algebra; concrete algebra is applied algebra; and if we insist on abstracting further from ordinary algebra, we lose important mathematical properties, and have, as a result, not algebra but only certain generic features of algebra. Whereas, in the case of geometry, we lose no mathematical properties in passing from concrete to abstract geometry. The true parallelism holds, of course, between the relation of ordinary algebra to applied algebra and the relation of abstract geometry to concrete geometry. In passing from applied algebra to ordinary algebra, we do not lose any mathematical properties.

The fact upon which I have been insisting involves the author's procedure in certain important defects, notably the following. He writes down a set of axioms and proves that they are consistent. But it does not in the least follow that the corresponding algebraic propositions are consistent, since these propositions are more determinate logically. Similarly, he proves that his axioms are independent; but, for the same reason, it does not follow that the corresponding algebraic propositions are independent.

C. H. LANGFORD

The Mathematics of Finance. By H. W. KUHN and C. C. MORRIS. Boston, Houghton Mifflin Company, 1926. xiii+340 pages. Price $3.00.

This is an introductory book on the subject suitable for a first course. It covers more ground than most of the current treatises of this nature and one notices in turning over the leaves that 104 out of the 257 pages of subject matter are devoted to the study of life annuities and life insurances. The tables in the

back of the book are more than usually adequate for class-room use. They include a six place table of logarithms, with proportional parts, for numbers from one to ten thousand. Answers are given to most of the problems but not to all of them.

The authors' policy of meting out large doses of the theory in one chapter and following this by a chapter consisting only of applications can be criticized from the pedagogical standpoint. However, this is perhaps more than compensated for by their frequent use of graphs and by the fact that many of the conventional formulae which must abound in such a treatise are followed by theorems which state in words the import of the formulae. Moreover the number of formulae to be memorized is somewhat reduced, it being shown explicitly that many auxiliary formulae are simply special cases of a general formula.

The statements and proofs of the theorems are precise and the wording of the problems leaves no room for ambiguity. The brief treatment of the theory of probability is rather better than the usual treatment in such texts.

B. F. KIMBALL

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin of at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3305. Proposed by Paul Wernicke, Washington, D.C.

(a) Prove that x^2+y^2, where x and y are integers, cannot be the square of an integer, unless x or y is divisible by 3.

(b) Modify the theorem for the case that both x and y are divisible by 3.

(c) Generalize the theorem so as to cover other exponents thereby proving a part of Fermat's greater theorem.

3306. Proposed by H. A. Mangan, Vandalia Public Schools, Mo.

A man bought a horse for A dollars and agreed to pay interest and principal in n equal monthly installments, interest r percent per annum. Show how to compute r and n.

As a practical example, a certain loan company will lend \$1200 to be repaid in 120 monthly installments of \$12.50 each. What is the rate of interest? Show how to compute it.

3307. Proposed by C. N. Mills, Normal, Illinois.

A pyramid whose faces are equilateral triangles and whose base is a square stands on a horizontal plane and faces the cardinal points. If α is the sun's altitude and β the distance from the meridian and θ the vertical angle of the shadow prove that

$$\tan\theta = \tan 2\alpha \cos(\tfrac{1}{4}\pi - \beta).$$

3308. Proposed by Victor d'Unger, Little Rock, Arkansas.

Given an ellipse, $a^{-2}x^2+b^{-2}y^2=1$, where a and b are the semi-axes; it is required to erect n perpendiculars to the major axis such that they intercept equal lengths of arc on the ellipse. What will be the distances between these perpendiculars along the major axis?

3309. Proposed by Otto Dunkel, Washington University.

If (1, 2, 3, 4, 5, 6) denotes the determinant of the 6th order whose ith row is

$$x_i^2,\ x_iy_i,\ y_i^2,\ x_i,\ y_i,\ 1, \quad \text{and if } (i,j,k) = \begin{vmatrix} x_i, & y_i, & 1 \\ x_j, & x_j, & 1 \\ x_k, & y_k, & 1 \end{vmatrix},$$

show that

$$(1,2,3,4,5,6) = (6,1,2)(2,3,4)(4,5,6)(1,3,5) - (1,2,3)(3,4,5)(5,6,1)(2,4,6).$$

is an identity in the 12 independent variables x_i, y_i.

UNSOLVED PROBLEMS

Readers are requested to send in solutions for the following unsolved problems which were proposed in the years 1918, 1919, 1920. The number of each problem is printed in bold face type, with the page number following.

1918

2663, 19; **2686,** 118; **2695,** 170; **2713,** 259; **2717,** 260; (1925, p. 316); **2721,** 303; **2725,** 303; **2727,** 396; **2730,** 397.

1919

2739, 35; **2743,** 36; **2762,** 170 (1925, p. 434); **2764,** 171; **2775,** 213 (1925, p. 520); **2778,** 268; **2786,**366; **2787,** 366; **2788,** 413; **2798,** 458. Note. Problem 2764 is incorrectly numbered 2740.

1920

2807, 80; **2808,** 80; **2810,** 81; **2817,** 134; **2821,** 134; (1925, p. 481); **2826,** 186; **2837,** 273; **2844,** 326, **2847,** 327; **2848,** 327; **2849,** 327; **2854,** 377; **2856,** 377; **2867,** 482.

SOLUTIONS

3244 [1927, 156]. Proposed by V. M. Spunar, Chicago, Illinois.

Divide a triangle by two straight lines into three parts, which when properly arranged shall form a parallelogram whose angles are of given magnitudes.

Solution by Harry C. Bradley, Mass. Inst. of Technology

Divide the triangle into two parts by a line parallel to the base and bisecting each of the other two sides. Rotate the triangular piece cut off through 180°, and place against the quadrilateral piece so as to form a parallelogram. Make a cut across the quadrilateral piece at the required angle with the sides of this parallelogram. Transfer the piece cut off from one end of the parallelogram to the other, thus forming a second parallelogram having the required angles.

With certain triangles and given angles a solution in three pieces may not always be possible, since the second cut must be confined wholly within the quadrilateral piece left by the first cut. But there are, in general, three chances, as any side of the given triangle may be taken as the base.

Also solved by ALICE ANN GRANT, MICHAEL GOLDBERG, and A. PELLETIER.

3245 [1927, 156]. Proposed by A. A. Bennett, Lehigh University.

Let $r=f(x+y+z)$, $s=f(x-y-z)$, $t=f(-x+y-z)$, $u=f(-x-y+z)$. Obtain in symmetrical form with as little computation as possible the algebraic identity in r,s,t,u, for each of the three cases, $f(x)=\sin x$, $\cos x$, $\tan x$.

SOLUTION BY J. A. BULLARD, U. S. Naval Academy

Since the sum of the four angles is zero, we have, in particular,

$$\sin[(x+y+z)+(x-y-z)] = -\sin[(-x+y-z)+(-x-y+z)],$$

which on expansion gives

$$r(1-s^2)^{1/2}+s(1-r^2)^{1/2}+t(1-u^2)^{1/2}+u(1-t^2)^{1/2}=0,$$

for $f(x)=\sin x$. Similarly,

$$s(1-t^2)^{1/2}+t(1-s^2)^{1/2}+r(1-u^2)^{1/2}+u(1-r^2)^{1/2}=0,$$
$$s(1-u^2)^{1/2}+u(1-s^2)^{1/2}+r(1-t^2)^{1/2}+t(1-r^2)^{1/2}=0.$$

Adding these equations we have

$$(1-r^2)^{1/2}(s+u+t)+(1-s^2)^{1/2}(r+u+t)+(1-t^2)^{1/2}(r+s+u)+(1-u^2)^{1/2}(r+s+t)=0.$$

If $f(x)=\cos x$, we have similarly

$$rs-(1-r^2)^{1/2}(1-s^2)^{1/2}=tu-(1-t^2)^{1/2}(1-u^2)^{1/2},$$
$$st-(1-s^2)^{1/2}(1-t^2)^{1/2}=ru-(1-r^2)^{1/2}(1-u^2)^{1/2},$$
$$su-(1-s^2)^{1/2}(1-u^2)^{1/2}=rt-(1-r^2)^{1/2}(1-t^2)^{1/2}.$$

If we rearrange terms, square, and add, we get

$$3(r^2+s^2+t^2+u^2)-6rstu\pm 6[(1-r^2)(1-s^2)(1-t^2)(1-u^2)]^{1/2}=6,$$

where the square root has the sign of the product of the sines of the angles involved.
Again, if $f(x)=\tan x$,

$$f[(x+y+z)+(x-y-z)] = -f[(-x+y-z)+(-x+y+z)],$$

and

$$(r+s)/(1-rs) = -(t+u)/(1-tu),$$

or $r+s+t+u=rst+rsu+rtu+stu$.

Also solved by MICHAEL GOLDBERG.

3246 [1927, 156]. Proposed by Daniel Kreth, Wellman, Iowa.

From a point within a square, lines drawn to three of the vertices are 30, 40, and 50. Required the side of the square. A geometrical solution is desired.

SOLUTION BY SMITH D. TURNER, Cambridge, Mass.

Let x be the side of the square with vertices A, C, D, B in counter clockwise order, and P a point such that $PA=a$, $PB=b$, $PC=c$ are given distances. Rotate in the plane of the figure the triangle APB through 270° about A and thus obtain a quadrilateral $PAP'C$ with $\angle PAP'=90°$ and the diagonal $PP'=a\sqrt{2}$. The second diagonal x will now be determined. The area A of a quadrilateral with sides d, e, f, g, in order, and diagonals u, v, is given by

$$16A^2 = 4u^2v^2-(d^2+f^2-e^2-g^2)^2.*$$

* See the article *The Area of a Quadrilateral* by R. C. Archibald in this Monthly, vol. 29 (1922), p. 32, for references to this formula.

Applying this to $PAP'C=\Delta PAP'+\Delta P'CP$, we have

$$[s(s-a\sqrt{2})(s-b)(s-c)]^{1/2}\pm\frac{a^2}{2}=\frac{1}{4}[8a^2x^2-b^4-c^4+2b^2c^2]^{1/2},$$

where $s=(a\sqrt{2}+b+c)/2$. Hence

$$x=\left[\frac{b^2+c^2\pm(4a^2b^2+4a^2c^2+2b^2c^2-4a^4-b^4-c^4)^{1/2}}{2}\right]^{1/2},$$

where the $+$ sign is used when P is within the square, and the $-$ sign when P is without. For $a=30$, $b=40$, $c=50$,

$$x=10\left(\frac{41\pm\sqrt{1071}}{2}\right)^{1/2}=60.7,\quad 20.4.$$

Also solved by NATHAN ALTSHILLER-COURT, H. C. BRADLEY, S. A. COREY, L. R. CHASE, L. A. DYE, MICHAEL GOLDBERG, DANIEL KRETH, R. M. MCFARLAND, A. PELLETIER, and PAUL WERNICKE.

3248 [1927, 156]. Proposed by F. A. Foraker, University of Pittsburgh.

Construct a triangle, given the sum of two sides, the third side, and the bisector of the angle opposite the third side.

SOLUTION BY PAUL WERNICKE, Washington, D. C.

In triangle ABC let the sides a, b, c be divided by the bisectors of the angles, t_a, t_b, t_c into the parts a', a''; b', b''; c', c'', respectively, the arrangement being counterclockwise throughout.

GIVEN: a, $b+c$, t_a.

ANALYSIS: We know that $b:a'=(b+c):a$. Calling F the foot of t_a, bisector t_a in triangle ACF will meet the side $t_a=AF$ in a pcint M dividing AF in the known ratio $b:a'=b+c:a$. This determines the intersection, M, of the bisectors t_a, t_b, t_c.

CONSTRUCTION: On a line lay off $AF=t_a$. Divide AF internally at M and externally at N in the ratio $b+c:a$. Draw a circle with the diameter MN. This is a locus of both B and C. Through F draw a chord $BC=a$. ABC will be the required triangle, to be obtained, of course, in two positions.

DELIMITATION: In order that a may be a chord through F, we must have

$$\frac{2a(b+c)t_a}{(b+c)^2-a^2}>a>\frac{2at_a}{[(b+c)^2-a^2)]^{1/2}},$$

the greater quantity on the left being the diameter MN, and the smaller one on the right, a chord perpendicular to MN at F. We may write,

$$2(b+c)t_a>(b+c)^2-a^2>2t_a[(b+c)^2-a^2]^{1/2}$$

Also solved by NATHAN ALTSHILLER-COURT, THEODORE BENNETT, A. PELLETIER, and J. Q. MCNATT.

3249[1927, 156]. Proposed by Otto Dunkel, Washington University.

If $i!f_i=d^if(x)/dx^i$ and

$$D=\begin{vmatrix} f_r & f_{r+1}\cdots f_{r+n-1} \\ f_{r+1} & f_{r+2}\cdots f_{r+n} \\ \cdot & \cdot\quad\cdot \\ \cdot & \cdot\quad\cdot \\ \cdot & \cdot\quad\cdot \\ f_{r+n-1} & f_{r+n}\cdots f_{r+2n-2} \end{vmatrix},\quad \text{prove that } \frac{dD}{dx}=(r+2n-1)D',$$

where D' is the determinant D with the subscripts of the last row increased by unity.

SOLUTION

Set $a_{ij}=f_{r+i+j-2}$, $(j=1, 2, \cdots, n,\ i=1, 2, \cdots, n+1)$.

Then $D=\sum_{j=1}^{j=n} a_{ij}A_{ij}$ $(i=1, 2, \cdots, n)$ where A_{ij} denotes the cofactor of a_{ij} in D; also $da_{ij}/dx=(r+i+j-1)a_{i+1,j}$. Hence

$$\frac{dD}{dx}=\sum_{i=1}^{i=n}\left[\sum_{j=1}^{j=n}(r+i+j-1)a_{i+1,j}A_{ij}\right].$$

Subtract from each bracket above

$$(r+2i-1)\sum_{i=1}^{i=n}a_{i+1,j}A_{ij}=0, \quad i=1,2,\cdots,n-1$$

$$=(r+2n-1)D', \quad i=n,$$

and there results

$$\frac{dD}{dx}=\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}(j-i)a_{i+1,j}A_{ij}+(r+2n-1)D'$$

$$=(r+2n-1)D',$$

for to each term in the double sum there is a corresponding term $(i-j)a_{j+1,i}A_{ji}$, except where $i=j$, and the two cancel since $a_{i+1,j}=a_{j+1,i}$ and $A_{ij}=A_{ji}$.

Also solved by RICHARD S. BURINGTON and PAUL WERNICKE.

3252[1927, 217]. Proposed by the late Laenas G. Weld.

Find the equation of the plane curve along which a point, moving with a velocity in constant ratio to its ordinate, will pass from (x_1, y_1) to (x_2, y_2) in the least possible time.

SOLUTION BY RAYMOND GARVER, University of Rochester.

The integral to be minimized is obviously $I=\int_{x_1}^{x_2} f\,dx$, $f=(1+y'^2)^{1/2}y^{-1}$. We must have $y\neq 0$, and may agree to take y positive. To determine the extremals for this problem we may apply the condition $f-y'f_{y'}=$constant, (where subscripts represent formal partial differentiation), which follows from Euler's differential equation when f does not contain x explicitly, and when a second derivative exists along the minimizing arc. (See Bliss, *Calculus of Variations*, page 48). That a second derivative does exist in this case may be shown by a method exactly like that used by Bliss in his discussion of the brachistochrone problem (Bliss, page 50).

Since $f_{y'}=y'y^{-1}(1+y'^2)^{-1/2}$, the above condition reduces to

$$y^{-1}(1+y'^2)^{1/2}-y'^2y^{-1}(1+y'^2)^{-1/2}=y^{-1}(1+y'^2)^{-1/2}=c^{-1}$$

and the solution is $(x+k)^2+y^2=c^2$. The extremals are thus a two-parameter family of semi-circles (since $y>0$), with centers on the x-axis. One and only one circle of this family will pass through (x_1, y_1) and (x_2, y_2), if $x_1\neq x_2$ and this actually will be the minimizing arc. (If $x_1=x_2$, the minimizing arc will be a straight line.) To complete the discussion, sufficiency conditions must be applied. These are relatively simple when the problem is "regular." (See Bliss, pages 154 and 159). The Jacobi condition is satisfied here; the test given in Bliss, 149, may be applied. And if we cover the half-plane in which y is positive by the one-parameter family of extremals obtained by putting $k=k_1$, we have a field (in the Weierstrass sense) in which $f_{y'y'}=y^{-1}(1+y'^2)^{-3/2}>0$ for $y>0$. These form sufficient conditions that the semi-circle actually be a minimizing arc.

Also solved by MICHAEL GOLDBERG and F. L. WILMER.

3254[1927, 217]. Proposed by C. M. Sparrow, Rouss Physical Laboratory, University of Virginia.

ABC is a triangle with angles α, β, γ. Pairs of lines are drawn through the vertices, symmetrical with respect to the bisectors and making angles $\lambda\alpha, \lambda\beta, \lambda\gamma$ with the sides (λ any real multiple). The two

lines adjacent to any side determine a point. Prove that the three points so determined form a triangle which is perspective to the original triangle.

SOLUTION BY PAUL WERNICKE, Washington, D.C.

Over the base BC we have the given triangle with the sides $2R \sin C$, $2R \sin B$, where R is the radius of the circumscribed circle, and a second triangle $A'BC$ with the sides $2R' \sin \lambda C$, $2R' \sin \lambda B$. Taking the origin at B and the x-axis along BC, we have for the coördinates of A, $(2R \sin C \cos B, 2R \sin C \sin B)$ and for A', $(2R' \sin \lambda C \cos \lambda B, 2R' \sin \lambda C \sin \lambda B)$. Writing the equation of the straight line AA' in determinant form and then setting $y=0$, we obtain the intercept BA_1 from

$$\begin{vmatrix} x & 2R \sin C \cos B & 2R' \sin \lambda C \cos \lambda B \\ 0 & 2R \sin C \sin B & 2R' \sin \lambda C \sin \lambda B \\ 1 & 1 & 1 \end{vmatrix} = 0$$

or

$$BA_1 (R \sin C \sin B - R' \sin \lambda C \sin \lambda B) = 2 RR' \sin C \sin \lambda C \sin (1 - \lambda) B .$$

If in this result we interchange C and B we obtain A_1C. Hence

$$BA_1/A_1C = \sin C \sin \lambda C \sin (1 - \lambda)B / \sin B \sin \lambda B \sin (1 - \lambda)C .$$

Multiplying the three ratios thus obtained for the three sides, we evidently obtain unity for the product. Hence by Ceva's theorem the three lines AA', BB', CC' are concurrent.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Miss ALICE K. BELL is studying at the University of Michigan while on leave of absence from the State Teachers College, Fresno, California.

Professor C. M. CONWELL of Albany State College for Teachers has resigned to accept a mastership at St. Paul's School, Concord, N.H.

Mr. B. F. DOSTAL of the University of Michigan has been appointed assistant professor of mathematics at the University of Florida.

Dr. LESTER S. HILL has been appointed assistant professor of mathematics at Hunter College.

Professor R. L. JEFFREY of Acadia University is completing work for his doctorate at Cornell University.

Professor GERALD E. MOORE of Hanover College is completing work for his doctorate at the University of Illinois.

Miss MARGUERITE REINERT, formerly instructor at the University of California at Los Angeles, is now instructor at Pasadena Junior College.

Dr. GLENN F. ROUSE of Lehigh University has been appointed assistant professor of physics and mathematics at American University. Professor F. SHENTON is chairman of the department of mathematics.

Associate Professor LAO G. SIMONS has been promoted to a full professorship and head of the department of mathematics at Hunter College.

Mr. C. W. STROM of Luther College returned this year from three years of study at Oxford University.

Dr. D. V. WIDDER has been appointed associate professor of mathematics at Bryn Mawr College.

Professor F. E. WOOD of Northwestern University is spending the year in research in Bologna and Rome.

Dr. H. S. WALL and Mr. K. G. FULLER have been appointed instructors in mathematics at Northwestern University.

The following courses in mathematics are announced for the summer of 1928:

University of Chicago, first term, June 18 to July 25; second term, July 26 to August 31. In addition to the usual courses in trigonometry, plane analytical geometry and differential and integral calculus, the following advanced courses are announced. By Professor L. E. DICKSON: Topics in the theory of numbers. By Professor E. T. BELL: Theory of functions of a complex variable; Applications of analysis to the arithmetic of quadratic forms. By Professor E. P. LANE: Synthetic projective geometry; Projective differential geometry I. By Professor F. D. MURNAGHAN: Modern hydrodynamical theory. By Professor M. I. LOGSDON: Solid analytic geometry; Higher plane curves. By Professor L. M. GRAVES: Modern theories of integration. By Professor R. W. BARNARD: Advanced calculus; Introduction to higher algebra. By Professor WALTER BARTKY: Introduction to celestial mechanics II. By Dr. F. R. BAMFORTH: Differential equations.

University of Colorado, first term, June 18 to July 21; second term, July 23 to August 24. In addition to the usual elementary work in algebra, trigonometry, analytic geometry, calculus, the following courses will be offered. First term—By Professor LIGHT: Teacher's course in mathematics; History of mathematics; Calculus of variations. By Professor KEMPNER: Advanced teacher's course in mathematics; Differential equations; Group theory and introduction to Galois theory. Second term—By Professor LIGHT: Statistics; Theory of equations; Calculus of variations (continued). By Professor KEMPNER; Advanced teacher's course (repeated); Differential equations (continued); Group theory and introduction to Galois theory (continued).

Columbia University, July 9 to August 17. In addition to courses in trigonometry, solid geometry, college algebra, analytic geometry, and calculus, and a series of courses for teachers of secondary mathematics, the following advanced courses are offered. By Professor W. B. FITE: Differential equations; Theory of

infinite series. By Professor J. F. RITT: Theory of functions of a complex variable. By Professor K. W. LAMSON: Differential geometry. By Dr. B. O. KOOPMAN: Introduction to modern geometry.

University of Illinois, June 18 to August 11.[1] In addition to the usual courses in college albegra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered. By Professor G. A. MILLER: Introduction to higher algebra, based on Bôcher; Critical study of the history of mathematics. By Professor J. B. SHAW: Functions of a real variable. By Professor H. R. BRAHANA: Projective geometry. By Dr. L. L. STEIMLEY: Differential equations. By Dr. H. W. BAILEY: Teachers' courses in elementary mathematics. By Mr. G. L. EDGETT: Mathematical statistics for teachers.

University of Iowa, first term, June 7 to July 20—In addition to courses in algebra, solid geometry, trigonometry, and calculus, the following subjects are offered. By Dr. M. A. NORDGAARD: Subject matter and teaching of mathematics. By Dr. CONKWRIGHT: Ordinary differential equations; Theory of functions of a complex variable. By Professor WYLIE: Elementary mechanics; Mathematics of finance; Descriptive astronomy. By Professor WOODS: Advanced coordinate geometry; Elliptic integrals. By Professor REILLY: Algebra for high school teachers; Disability theory applied to life insurance. By Professor CHITTENDEN: Advanced calculus; Functions of infinitely many variables. Second term, July 23 to August 24—By Dr. NORDGAARD: The history of mathematics. By Dr. WARD: Elementary mechanics; Theory of functions of a complex variable. By Professor BAKER: Differental equations; Geometry of forces. By Professor REILLY: The numerical solution of equations; Frequency distributions and correlations.

Johns Hopkins University, June 25 to August 3. In addition to elementary courses the following graduate course will be given. By Professor J. R. MUSSELMAN: Theory of algebraic equations and forms.

University of Kansas, first term,[1] June 5 to July 20—By Professor C. H. ASHTON: Modern synthetic geometry; Series. By Professor E. B. STOUFFER: Seminar. By Professor G. W. SMITH: Higher plane curves; College algebra. By Professor J. J. WHEELER: Differential calculus; Analytical geometry. Second term, July 23 to August 17—By Professor H. E. JORDAN: Integral calculus; Plane trigonometry. By Professor G. W. SMITH: Analytical mechanics; Seminar.

University of Nebraska, June 4 to July 13. In addition to the usual introductory courses up to and including the calculus, the following courses will be offered. By Professor BRENKE: Differential equations. By Professor GABA: Advanced Euclidean geometry; Foundations of algebra and geometry. By

Professor CAMP: Advanced algebra. By Professor PIERCE: Theory of equations. By Mr. HARPER: Elements of the mathematical theory of statistics.

Ohio State University, June 18 to August 31. In addition to the usual courses in trigonometry, analytic geometry, and calculus, the following courses are offered. By Professor C. L. ARNOLD: The teaching of mathematics; Differential equations. By Professor C. C. MORRIS: Theory of equations; Mathematical statistics. By Professor J. H. WEAVER: Differential geometry; Introduction to higher algebra.

University of Pennsylvania, July 2 to August 11. In addition to courses in trigonometry, college algebra, analytic geometry, and calculus, the following courses are offered. By Professor H. H. MITCHELL: Probability. By Professor M. J. BABB; College geometry. By Professor ARNOLD DRESDEN, of Swarthmore College: Elliptic functions; Differential equations. By Professor J. R. KLINE: Theories of integration. By Professor J. M. THOMAS: Introduction to tensor analysis.

University of Wisconsin, six weeks session, June 25 to August 3. By Professor J. H. TAYLOR: Differential equations; Differential geometry. By Professor E. B. VAN VLECK: Modern geometrical concepts; The location of the roots of an algebraic equation. Special nine weeks course for graduates, June 25 to August 24. By Professor R. E. LANGER: Partial differential equations; Calculus of variations. By Professor WARREN WEAVER: Theory of potential; Introduction to statistical mechanics.

Professor F. J. DICK, of the Theosophical University, Point Loma, died May 25, 1927.

Captain D. M. GARRISON, head of the department of mathematics at St. John's College, Annapolis, Md., died suddenly on December 30, 1927. He had been a member of the Association since 1923.

Mr. RAYMOND L. MODESITT of the Eastern Illinois State Teachers College, a member of the Association since September, 1921, died suddenly on December 16, 1927.

Professor H. E. RUSSELL, of the University of Denver, died May 31, 1927.

CORRIGENDA

The following corrections should be made in the December (1927) issue of the Monthly:

On page 499, in line 21, replace the phrase "a branch of the American Academy of Science" by "The Louisiana Academy of Sciences."

On page 518, in line 25 and in line 26, replace z by r.

THE TWELFTH ANNUAL MEETING OF THE ASSOCIATION

The twelfth annual meeting of the Mathematical Association of America was held at Nashville, Tenn., on Thursday and Friday, December 29–30, 1927 in conjunction with the annual meeting of the American Mathematical Society and in affiliation with the American Association for the Advancement of Science. Two hundred twenty were in attendance at these meetings, among them the following one hundred sixty-seven members of the Association.

O. P. AKERS, Allegheny College.
BERD R. ALLEN, Galloway College.
FLORENCE E. ALLEN, University of Wisconsin.
R. B. ALLEN, Kenyon College.
MARY ANDERSON, Illinois Woman's College.
ETHEL L. ANDERTON, Smith College.
R. C. ARCHIBALD, Brown University.
P. L. ARMSTRONG, West Tennessee Teachers College.
C. H. ASHTON, University of Kansas.
C. S. ATCHISON, Washington and Jefferson College.

C. A. BARNHART, University of New Mexico.
DAVID BARROW, University of Georgia.
W. S. BECKWITH, Georgia State Teachers College.
O. F. H. BERT, Washington and Jefferson College.
WILLIAM BETZ, East High School, Rochester, N. Y.
R. V. BLAIR, Vanderbilt University.
H. F. BLICHFELDT, Stanford University.
J. D. BOND, University of Tennessee.
L. H. BOWEN, Furman University.
P. P. BOYD, University of Kentucky.
E. W. BROWN, Yale University.
E. T. BROWNE, University of North Carolina.
H. E. BUCHANAN, Tulane University.
R. S. BURINGTON, Case School of Applied Science.

W. D. CAIRNS, Oberlin College.
R. D. CARMICHAEL, University of Illinois.
I. S. CARROLL, Syracuse University.
J. B. COLEMAN, University of South Carolina.
JULIA T. COLPITTS, Iowa State College.
J. L. COOLIDGE, Harvard University.
LENNIE P. COPELAND, Wellesley College.
A. R. CRATHORNE, University of Illinois.

JULIA DALE, Mississippi Delta State College.
H. T. DAVIS, Indiana University.
MARY S. DAY, Salem College.
L. A. V. DE CLEENE, St. Norbert's College.
W. W. DENTON, University of Michigan.
ELEANOR C. DOAK, Mount Holyoke College.
B. F. DOSTAL, University of Florida.
ARNOLD DRESDEN, Swarthmore College.

W. W. ELLIOTT, Duke University.
G. C. EVANS, Rice Institute.
G. W. EVANS, Lynn, Mass.

B. F. FINKEL, Drury College.
L. R. FORD, Rice Institute.
W. B. FORD, University of Michigan.
TOMLINSON FORT, Lehigh University.

C. A. GARABEDIAN, University of Cincinnati.
H. M. GEHMAN, Yale University.
EMMA M. GIBSON, High School, Springfield, Mo.
J. L. GIBSON, University of Utah.
J. S. GOLD, Bucknell University.

J. A. HARDIN, Centenary College.
MARGARET E. HARRIS, Grenada College.
E. R. HEDRICK, University of California at Los Angeles.
ARCHIBALD HENDERSON, University of North Carolina.
GERTRUDE A. HERR, Iowa State College.
G. W. HESS, Howard College.
R. E. HILL, University of Louisville.
F. J. HOLDER, Mercer University.
H. M. HOSFORD, Southern Methodist University.
ANNA M. HOWE, Newcomb Memorial College.
JEWELL C. HUGHES, University of Arkansas.
E. V. HUNTINGTON, Harvard University.
MABEL HUTCHINS, Blue Mountain College.
EMMA HYDE, Kansas State Agricultural College.

LOUIS INGOLD, University of Missouri.
M. H. INGRAHAM, University of Wisconsin.

DUNHAM JACKSON, University of Minnesota.
E. H. JONES, Southern Methodist University.
S. I. JONES, Life and Casualty Ins. Co., Nashville.